Zeta-functions of renormalizable sub-Lorenz templates

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Abstract

We describe the Williams zeta functions and the twist zeta functions of sub-Lorenz templates generated by renormalizable Lorenz maps, in terms of the corresponding zeta-functions of the sub-Lorenz templates generated by the renormalized map and by the map that determines the renormalization type.

1 Introduction

Let ϕ_t be a flow on S^3 with countably many periodic orbits $(\tau_n)_{n=1}^{\infty}$. We can look to each closed orbit as a knot in S^3. It was R. F. Williams, in 1976, who first conjectured that non trivial knotting occur in the Lorenz system ([17]). In 1983, Birmann and Williams introduced the notion of template, in order to study the knots and links (i.e. finite collections of knots, taking into account the knotting between them) contained in the geometric Lorenz attractor ([2]).

A template, or knot holder, consists of a branched two manifold with charts of two specific types, joining and splitting, together with

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an expanding semiflow defined on it, see Figure 1. The relationship between templates and links of closed orbits in three dimensional flows is expressed in the following result, known as Template Theorem, due to Birman and Williams in [2].

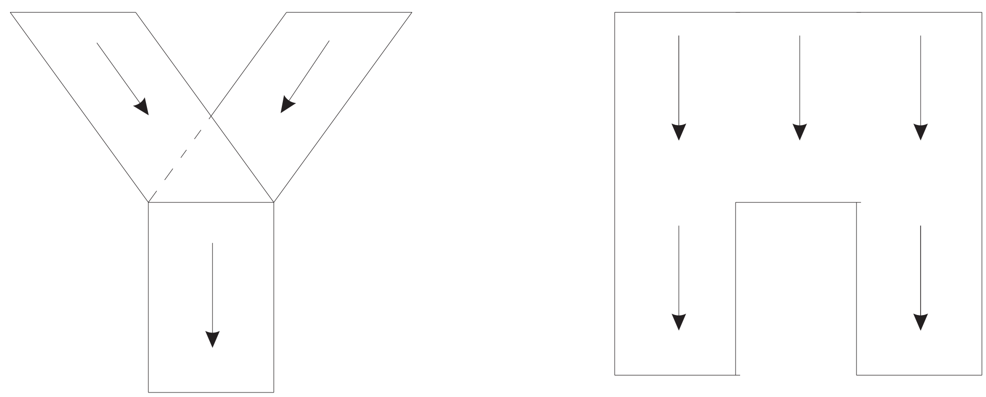

Figure 1: Charts of templates: joining (left) and splitting (right)

Theorem 1 *Given a flow ϕ_t on a three-manifold M, having a hyperbolic chain-recurrent set, the link of closed orbits L_ϕ is in bijective correspondence with the link of closed orbits $L_{\mathcal{T}}$ on a particular embedded template $\mathcal{T} \subset M$. On any finite sublink, this correspondence is via ambient isotopy.*

We define a *Lorenz flow* as a semi-flow that has a singularity of saddle type with a one-dimensional unstable manifold and an infinite set of hyperbolic periodic orbits, whose closure contains the saddle point (see [10]). A Lorenz flow, together with an extra geometric assumption is called a *geometric Lorenz flow* (see [18]). The dynamics of this type of flows can be described by the iteration of one-dimensional first-return maps $f : [a,b] \setminus \{c\} \to [a,b]$ with one discontinuity at $c \in]a,b[$, strictly increasing in the continuity intervals $[a,c[$ and $]c,b]$ and boundary anchored (i.e. $f(a) = a$ and $f(b) = b$), see [10]. These maps are called Lorenz maps and sometimes we denote them by $f = (f_-, f_+)$, where f_- and f_+ correspond, respectively, to the left and right branches.

In [7], Holmes studied families of iterated horseshoe knots which arise naturally associated to sequences of period-doubling bifurcations of unimodal maps.

It is well known, see for example [4], that period doubling bifurcations in the unimodal family are directly related with the creation of a 2-renormalization interval, i.e. a subinterval $J \subset I$ containing the critical point, such that $f^2|_J$ is unimodal.

Basically there are two types of bifurcations in Lorenz maps (see [13]): one is the usual saddle-node or tangent bifurcations, when the graph of f^n is tangent to the diagonal $y = x$, and one attractive and one repulsive n-periodic orbits are created or destroyed; the others are homoclinic bifurcations, when $f^n(0^{\pm}) = f^{n-1}(f_{\pm}(0)) = 0$ and one attractive n-periodic orbit is created or destroyed in this way, these bifurcations are directly related with homoclinic bifurcations of flows modelled by this kind of maps (see [13]).

Considering a monotone family of Lorenz maps (see [10]), the homoclinic bifurcations are realized in some lines in the parameters space, called *hom-lines* or *bifurcation bones*.

It is known that (see [13] and [9]), in the context of Lorenz maps, renormalization intervals are created in each intersection of two hom-lines. These points are called *homoclinic points* and are responsible for the self-similar structure of the bifurcation skeleton of monotone families of Lorenz maps. So it is reasonable to say that *homoclinic points are the Lorenz version of period-doubling bifurcation points*.

In [16], Williams considered Lorenz maps with a double saddle connection, i.e., such that $f^n(0^{\pm}) = f^{n-1}(f_{\pm}(0)) = 0$, and introduced a new determinant-like invariant to classify templates generated by them, in [5] these templates are called *sub-Lorenz templates* (note that, in a monotone family of Lorenz maps, maps corresponding to homoclinic points have a double saddle connection). This invariant is a zeta-function that counts periodic orbits taking into account their knot type and the linking between them. In [14], Michael Sullivan introduced one other zeta function that counts periodic orbits in positive templates, taking into account the number of twists in their stable manifolds.

In this paper we will study the effect of renormalization over these two invariants, in the context of sub-Lorenz templates.

2 Symbolic dynamics of Lorenz maps

Symbolic dynamics is a very useful combinatoric tool to study the dynamics of one-dimensional maps.

Let $f^j = f \circ f^{j-1}$, $f^0 = id$, be the j-th iterate of the map f. We define the *itinerary* of a point x under a Lorenz map f as $i_f(x) =$

$(i_f(x))_j$, $j = 0, 1, \ldots$, where

$$(i_f(x))_j = \begin{cases} L & \text{if} \quad f^j(x) < 0 \\ 0 & \text{if} \quad f^j(x) = 0 \\ R & \text{if} \quad f^j(x) > 0 \end{cases}.$$

It is obvious that the itinerary of a point x will be a finite sequence in the symbols L and R with 0 as its last symbol, if and only if x is a pre-image of 0 and otherwise it is one infinite sequence in the symbols L and R. So we consider the symbolic space Σ of sequences $X_0 \cdots X_n$ on the symbols $\{L, 0, R\}$, such that $X_i \neq 0$ for all $i < n$ and: $n = \infty$ or $X_n = 0$, with the lexicographic order relation induced by $L < 0 < R$.

It is straightforward to verify that, for all $x, y \in [-1, 1]$, we have the following:

1. If $x < y$ then $i_f(x) \leq i_f(y)$, and
2. If $i_f(x) < i_f(y)$ then $x < y$.

We define the *kneading invariant* associated to a Lorenz map $f = (f_-, f_+)$, as

$$K_f = (K_f^-, K_f^+) = (L i_f(f_-(0)), R i_f(f_+(0))).$$

We say that a pair $(X, Y) \in \Sigma \times \Sigma$ is *admissible* if $(X, Y) = K_f$ for some Lorenz map f. Denote by Σ^+, the set of all admissible pairs.

Consider the *shift map* $s : \Sigma \setminus \{0\} \to \Sigma$, $s(X_0 \cdots X_n) = X_1 \cdots X_n$. The set of admissible pairs is characterized, combinatorially, in the following way (see [9]).

Proposition 1 *Let $(X, Y) \in \Sigma \times \Sigma$, then $(X, Y) \in \Sigma^+$ if and only if $X_0 = L$, $Y_0 = R$ and, for $Z \in \{X, Y\}$ we have:*
(1) If $Z_i = L$ then $s^i(Z) \leq X$;
(2) If $Z_i = R$ then $s^i(Z) \geq Y$; with inequality (1) (resp. (2)) strict if X (resp. Y) is finite.

On the other hand, the kneading invariant of a map characterizes completely its combinatorics, more precisely, considering a pair $(X, Y) \in \Sigma^+$, denote by $\Sigma^+(X, Y)$ the set of sequences $Z \in \Sigma$ that satisfy conditions (1) and (2) from the previous proposition, then we have the following proposition whose proof can be found in [9].

Proposition 2 *Let $X \in \Sigma$ and f be a Lorenz map, then there exists $x \in I$ such that $X = i_f(x)$ if and only if $X \in \Sigma^+(K(f))$.*

2.1 Renormalization and $*$-product

In the context of Lorenz maps, we define renormalizability on the following way, see for example [13]:

Definition 1 *Let f be a Lorenz map, then we say that f is renormalizable if there exist $n, m \in \mathbb{N}$ with $n+m \geq 3$ and points $P < y_L < 0 < y_R < Q$ such that*

$$g(x) = \begin{cases} f^n(x) & if\, y_L \leq x < 0 \\ f^m(x) & if\, 0 < x \leq y_R \end{cases}$$

is a Lorenz map.

The map $R_{(n,m)}(f) = g = (f^n, f^m)|_{[y_L, y_R]}$ is called the (n, m)-renormalization of f.

Let $|X|$ be the length of a finite sequence $X = X_0 \cdots X_{|X|-1}0$, it is reasonable to identify each finite sequence $X_0 \cdots X_{|X|-1}0$ with the corresponding infinite periodic sequence $(X_0 \cdots X_{|X|-1})^\infty$, this is the case, for example, when we talk about the knot associated to a finite sequence.

Denote $\overline{X} = X_0 \ldots X_{|X|-1}$.

It is easy to prove that a pair of finite sequences

$$(X_0 \ldots X_{|X|-1}0, Y_0 \ldots Y_{|Y|-1}0)$$

is admissible if and only if the pair of infinite periodic sequences

$$((\overline{X})^\infty, (\overline{Y})^\infty)$$

is admissible.

We define the $*$-product between a pair of finite sequences $(X, Y) \in \Sigma \times \Sigma$, and a sequence $U \in \Sigma$ as

$$(X, Y) * U = \overline{U}_0 \overline{U}_1 \cdots \overline{U}_{|U|-1} 0,$$

where

$$\overline{U}_i = \begin{cases} \overline{X} & if\, U_i = L \\ \overline{Y} & if\, U_i = R \end{cases}.$$

Now we define the $*$-product between two pairs of sequences, $(X, Y), (U, T) \in \Sigma \times \Sigma$, X and Y finite, as

$$(X, Y) * (U, T) = ((X, Y) * U, (X, Y) * T).$$

The next theorem states that the reducibility relative to the $*$-product is equivalent to the renormalizability of the map. The proof can be found, for example, in [9].

Theorem 2 *Let f be a Lorenz map, then f is renormalizable with renormalization $R_{(n,m)}(f)$ iff there exist two admissible pairs (X,Y) and (U,T) such that $|X| = n$, $|Y| = m$, $K_f = (X,Y) * (U,T)$ and $K_{R_{(n,m)}(f)} = (U,T)$.*

We know from [9] that $(X,Y) * (U,T) \in \Sigma^+$ if and only if both $(X,Y) \in \Sigma^+$ and $(U,T) \in \Sigma^+$, so for each finite admissible pair (X,Y), the subspace $(X,Y)*\Sigma^+$ is isomorphic to the all space Σ^+, this provides a self-similar structure in the symbolic space of kneading invariants. It is straightforward to verify that the $*$-product of kneading invariants is associative, consequently this self-similar structure is nested. Now we will state a series of properties concerning the $*$-product, that are in the basis of our results.

The following lemma states that the order structure is reproduced at each level of renormalization.

Lemma 1 *Let (X,Y) be one admissible pair of finite sequences, and $Z < Z'$, then $(X,Y) * Z < (X,Y) * Z'$.*

The proof is straightforward.

For any sequence X and $0 \leq p < q < |X|$, we denote

$$X_{[p,q]} = X_p \ldots X_q.$$

Since we have an order structure in Σ, we will denote

$$[A,B] = \{X \in \Sigma : A \leq X \leq B\}.$$

For any finite sequence Z, consider also the numbers

$$n_L(Z) = \sharp \{i \leq |Z| - 1 : Z_i = L\}$$

and

$$n_R(Z) = \sharp \{i \leq |Z| - 1 : Z_i = R\}.$$

Lemma 2 *Let (X,Y) and (S,W) be admissible pairs and A and B be any two sequences in Σ such that $A \leq B$. Consider $Z \in \{X,Y\}$, then a sequence $K \in \Sigma \setminus \left\{Z_{[p,|Z|-1]}0\right\}$ belongs to $[Z_{[p,|Z|-1]}(X,Y)*A, Z_{[p,|Z|-1]}(X,Y)* B] \cap \Sigma^+((X,Y)*(S,W))$ if and only if $K = Z_{[p,|Z|-1]}(X,Y)*C$, with $C \in [A,B] \cap \Sigma^+((S,W))$.*

Proof

Obviously $K_{[0,|Z|-p-1]} = Z_{[p,|Z|-1]}$.

If $K_{|X|-p} = L$, then, since $K \geq Z_{[p,|Z|-1]}(X,Y) * A$, we have that $A_0 = L$ and $Z_{[p,|Z|-1]}(X,Y) * A = Z_{[p,|Z|-1]}X_{[0,|X|-1]}(X,Y) * \sigma(A)$, so $K_{[|X|-p,2|X]-p-1]} \geq X_{[0,|X|-1]}$.

On the other hand, since $K \in \Sigma^+((X,Y)*(S,W))$, then $K_{|X|-p} = L$ implies that $K_{[|X|-p,2|X]-p-1]} \leq X_{[0,|X|-1]}$.

Analogously we see that, if $K_{|X|-p} = R$ then $K_{[|X|-p,|X]+|Y|-p-1]} = Y_{[0,|Y|-1]}$.

Repeating these arguments ad infinitum we prove that $K = Z_{[p,|Z|-1]}(X,Y)* C$.

We will now prove that $C \in \Sigma^+((S,W))$.

Let us suppose by contradiction that $C \notin \Sigma^+((S,W))$. Then it happens one of the following situations:

(i) There exists l such that $C_l = L$ and $\sigma^l(C) > S$.

(ii) There exists l such that $C_l = R$ and $\sigma^l(C) < W$.

If it happens situation (i), then there exists r such that $C_{[l,l+r-1]} = S_{[0,r-1]}$ and $C_{l+r} > S_r$. But then, with $q = |Z|-p-1+|X|n_L(C_{[0,l-1]})+|Y|n_R(C_{[0,l-1]})$, we have that $K_q = X_0 = L$ and $\sigma^q(K) > (X,Y) * S$ and this implies that $K \notin \Sigma^+((X,Y) * (S,W))$.

If it happens situation (ii), we obtain the contradiction analogously

□

Lemma 3 *Let (X,Y) be one admissible pair of finite sequences,* $0 < q < |Y|$ *and* $Y_q = R$*, then* $Y_{[q,|Y|-1]}(X,Y) * Z \geq (\overline{Y})^\infty$*, for any sequence* Z*. Analogously, if* $0 < q < |X|$ *and* $X_q = L$*, then* $X_{[q,|X|-1]}(X,Y) * Z \leq (\overline{X})^\infty$*, for any sequence* Z*.*

Proof

Since (X,Y) is admissible, then $Y_{[q,|Y|-1]}(\overline{Y})^\infty > (\overline{Y})^\infty$, so there exists l such that $Y_{[q,q+l-1]} = Y_{[0,l-1]}$ and $Y_{q+l} > Y_l$. If $q+l < |Y|$ the result follows immediately. If $q+l \geq |Y|$, then necessarily $Y_{|Y|-q} = L$, because otherwise we would have $(\overline{Y})^\infty > Y_{|Y|-q}\ldots$ and $Y_{|Y|-q} = R$, and this violates admissibility. But then,

$$Y_{[|Y|-q,|Y|-1]}(\overline{Y})^\infty \leq (\overline{X})^\infty \leq (X,Y) * Z$$

and this gives the result. The proof of the second part is analogous.

□

Lemma 4 *Let (X, Y) be one admissible pair of finite sequences and $W, W' \in \{X, Y\}$. If $s^p((\overline{W})^\infty) < s^q((\overline{W'})^\infty)$ and $W_{[p,|W|-1]} \neq W'_{[q,|W'|-1]}$ then*

$$W_{[p,|W|-1]}(X, Y) * Z \leq W'_{[q,|W'|-1]}(X, Y) * Z'$$

for any sequences Z, Z'.

Proof The proof is divided in four cases: $W = X$ and $W' = Y$; $W = Y$ and $W' = X$; $W = W' = X$ and $W = W' = Y$. We will only demonstrate specifically the first case, since the others follow with analogous arguments.

Following the hypotheses, there exists l such that $X_{[p,(p+l-1) \mod |X|]} = Y_{[q,(q+l-1) \mod |Y|]}$ and $X_{(p+l) \mod |X|} < Y_{(q+l) \mod |Y|}$. If $l < \min\{|X| - p, |Y| - q\}$, then the result follows immediately.

If $|X|-p \leq |Y|-q$ and $X_{[p,|X|-1]} = Y_{[q,q+|X|-p-1]}$, then $Y_{q+|X|-p} = R$, because otherwise we would have $Y_{q+|X|-p} = L$ and $Y_{[q+|X|-p,|Y|-1]}Y^\infty > X^\infty$, and this violates admissibility of (X, Y). So $Y_{q+|X|-p} = R$ and, from Lemmas 1 and 3,

$$(X, Y) * Z \leq Y^\infty \leq Y_{[q+|X|-p,|Y|-1]}(X, Y) * Z', \tag{1}$$

and the result follows.

If $|X|-p \geq |Y|-q$ and $X_{[p,p+|Y|-q-1]} = Y_{[q,|Y|-1]}$, then $X_{p+|Y|-q} = L$, because otherwise we would have $X_{p+|Y|-q} = R$ and $X_{p+|Y|-q} \cdots < Y^\infty$, which contradicts admissibility of (X, Y). So $X_{p+|Y|-q} = L$ and

$$X_{[p+|Y|-q,|X|-1]}(X, Y) * Z \leq X^\infty \leq (X, Y) * Z' \tag{2}$$

and the result follows.

□

Let us now introduce some more notations:

$$m(A, B) = \min\left\{k \geq 0 : A_{|A|-1-k} \neq B_{|B|-k-1}\right\}.$$

$$\Sigma(A, B) = \{\sigma^n(A), \sigma^m(B) : 0 \leq n < |A|, 0 \leq m < |B|\},$$

and

$$\phi_{(A,B)} : \Sigma(A, B) \to \{1, \ldots, |A| + |B|\},$$

is the map that associates to each $X \in \Sigma(A, B)$, the position occupied by X in the lexicographic ordenation of $\Sigma(A, B)$.

For each $1 \leq k \leq |S|+|W|$, denote

$$I_k = \begin{cases} \left[(X,Y) * \phi^{-1}_{(S,W)}(k), (X,Y) * \phi^{-1}_{(S,W)}(k+1)\right] \text{ if } m(X,Y)=0 \\ \\ \left[X_{[|X|-m(X,Y),|X|-1]}(X,Y) * \phi^{-1}_{(S,W)}(k), X_{[|X|-m(X,Y),|X|-1]}(X,Y) * \phi^{-1}_{(S,W)}(k+1)\right] \\ \text{if } m(X,Y) \neq 0 \end{cases}$$

Remark 1 *From the previous three Lemmas we can take the following conclusions:*

1. *If $p < |X| - m(X,Y)$, denote by I_{X_p} the set $\left\{X_{[p,|X|-1]}(X,Y) * \sigma^k(Z) : Z \in \{S,W\} \text{ and } Z_{k-1} = L\right\}$. From Lemmas 1 and 2, $I_{X_p} = \left[X_{[p,|X|-1]}(X,Y) * \sigma^2(W), X_{[p,|X|-1]}(X,Y) * \sigma(S)\right] \cap \Sigma((X,Y) * (S,W))$, analogously, $p < |Y| - m(X,Y)$, denoting $I_{Y_p} = \left\{Y_{[p,|Y|-1]}(X,Y) * \sigma^k(Z) : Z \in \{S,W\} \text{ and } Z_{k-1} = R\right\}$ we have that $I_{Y_p} = \left[Y_{[p,|Y|-1]}(X,Y) * \sigma(W), Y_{[p,|Y|-1]}(X,Y) * \sigma^2(S)\right] \cap \Sigma((X,Y) * (S,W))$. On the other hand, if $p \geq |X| - m(X,Y)$, then $X_{[p,|X|-1]} = Y_{[q,|Y|-1]}$, where q is such that $|Y| - q = |X| - p$, and $\left[X_{[p,|X|-1]}(X,Y) * \sigma(W), X_{[p,|X|-1]}(X,Y) * \sigma(S)\right] \cap \Sigma((X,Y)*(S,W)) = \left\{X_{[p,|X|-1]}(X,Y)\sigma^k(Z) : Z \in \{S,W\}\right\}$. Without risk of confusion, we will denote these sets by I_{X_p}.*

2. *The ordenation of the elements of the sets I_{X_p} and I_{Y_q} is induced by the ordenation of the corresponding sequences $\sigma^k(Z)$ such that $Z \in \{S,W\}$. This follows immediately from Lemma 1.*

3. *For each $Z \in \{X,Y\}$, if $p \neq |Z| - m(X,Y) - 1$ then $\sigma(I_{Z_p}) = I_{Z_{p+1}}$. On the other hand, $\sigma(I_{X_{|X|-m(X,Y)-1}}) \cup \sigma(I_{Y_{|Y|-m(X,Y)-1}}) = I_{X_{|X|-m(X,Y)}}$. This follows immediately from the definitions.*

4. *Let $J_k = \left[\max I_{\phi^{-1}_{(X,Y)}(k)}, \min I_{\phi^{-1}_{(X,Y)}(k+1)}\right]$ and $H_k = \left[\phi^{-1}_{(X,Y)}(k), \phi^{-1}_{(X,Y)}(k+1)\right]$, it follows from Lemma 4 that $\sigma\left(J_{p_{k'}}\right) \subset J_{p_k}$ iff $\sigma\left(H_{k'}\right) \subset H_k$. Moreover, from (3) and Lemma 1, if $\phi^{-1}_{(X,Y)}(k) \notin \left\{\sigma^{|X|-m(X,Y)-1}(X), \sigma^{|Y|-m(X,Y)-1}(Y)\right\}$, then $\sigma\left(\max I_{\phi^{-1}_{(X,Y)}(k)}\right) = \max I_{\sigma\left(\phi^{-1}_{(X,Y)}(k)\right)}$ and $\sigma\left(\min I_{\phi^{-1}_{(X,Y)}(k)}\right) = \min I_{\sigma\left(\phi^{-1}_{(X,Y)}(k)\right)}$*

5. *Let $\mathcal{BL} = I_{|X|-m(X,Y)}$ if $m(X,Y) > 0$ and $\mathcal{BL} = I_{X_0} \cup I_{Y_0}$ if $m(X,Y) = 0$. From (2) and (3), performing some straightforward computations with the lengths of $(X,Y) * S$ and $(X,Y) * W$ we*

see that, for $k \neq k'$, then $\sigma^n(I_k) \cap \sigma^m(I_{k'}) \neq \emptyset$ if and only if both $\sigma^n(I_k)$ and $\sigma^m(I_{k'})$ are contained in $\mathcal{BL}$ and $\sigma^p(P_k) \cap \sigma^q(P_{k'}) \neq \emptyset$, where $P_k = \left[\phi^{-1}_{(S,W)}(k), \phi^{-1}_{(S,W)}(k+1)\right]$ and p and q are such that $n = |X| n_L \left(\phi^{-1}_{(S,W)}(k)_{[0,p-1]}\right) + |Y| n_R \left(\phi^{-1}_{(S,W)}(k)_{[0,p-1]}\right)$ and $m = |X| n_L \left(\phi^{-1}_{(S,W)}(k')_{[0,q-1]}\right) + |Y| n_R \left(\phi^{-1}_{(S,W)}(k')_{[0,q-1]}\right)$.

This remark is the kernel of the proofs of all our main results, so we will illustrate it with one example.

Example 1 *We will consider $(X,Y) = (LRRRL0, RLLR0)$ and $(S,W) = (LRR0, RL0)$. Denoting by $\underset{i}{Z} = \sigma^i(Z)$, for $Z \in \{X,Y,S,W\}$ and setting that $A < B$ if A is located at the left of B, we have the following ordenations of the members of the pairs (X,Y) and (S,W), moreover, the relative position of each member gives the maps $\phi_{(X,Y)}$ and $\phi_{(S,W)}$.*

$$\underset{1}{Y}\ \underset{4}{X}\ \underset{2}{Y}\ \underset{0}{X}\ \underset{0}{Y}\ \underset{3}{X}\ \underset{3}{Y}\ \underset{2}{X}\ \underset{1}{X}$$

$$\underset{1}{W}\ \underset{0}{S}\ \underset{0}{W}\ \underset{2}{S}\ \underset{1}{S}$$

We will now consider the $*$-product

$$(X,Y) * (S,W) = (\overline{\underset{0}{L}\underset{1}{R}\underset{2}{R}\underset{3}{R}\underset{4}{L}}^{S_0}\ \overline{\underset{5}{R}\underset{6}{L}\underset{7}{L}\underset{8}{R}}^{S_1}\ \overline{\underset{9}{R}\underset{10}{L}\underset{11}{L}\underset{12}{R}}^{S_2}0, \overline{\underset{0}{R}\underset{1}{L}\underset{2}{L}\underset{3}{R}}^{W_0}\ \overline{\underset{4}{L}\underset{5}{R}\underset{6}{R}\underset{7}{R}\underset{8}{L}}^{W_1}0),$$

we used the underscripts to indicate the corresponding iterate of the shift map and the upperscripts to indicate the element of (S,W) that generated each subword in the $*$-product. Denoting by $(A,B) = (X,Y) * (S,W)$ we will now order the elements of the $*$-product pair:

$$\overbrace{\underset{1}{B}\ \underset{10}{A}\ \underset{6}{A}}^{I_{Y_1}} \overset{J_1}{\longleftrightarrow} \overbrace{\underset{8}{B}\ \underset{4}{A}}^{I_{X_4}} \overset{J_2}{\longleftrightarrow} \overbrace{\underset{2}{B}\ \underset{11}{A}\ \underset{7}{A}}^{I_{Y_2}} \overset{J_3}{\longleftrightarrow} \overbrace{\underset{4}{B}\ \underset{0}{A}}^{I_{X_0}}\ \overbrace{\underset{0}{B}\ \underset{9}{A}\ \underset{5}{A}}^{I_{Y_0}} \overset{J_4}{\longleftrightarrow} \overbrace{\underset{7}{B}\ \underset{3}{A}}^{I_{X_3}} \overset{J_5}{\longleftrightarrow} \overbrace{\underset{3}{B}\ \underset{12}{A}\ \underset{8}{A}}^{I_{Y_3}} \overset{J_6}{\longleftrightarrow} \overbrace{\underset{6}{B}\ \underset{2}{A}}^{I_{X_2}} \overset{J_7}{\longleftrightarrow} \overbrace{\underset{5}{B}\ \underset{1}{A}}^{I_{X_1}}$$

as we can see, the ordered disposition of the members of $(X,Y)*(S,W)$ is obtained from the ordered disposition of (X,Y), inflating each $\underset{i}{Z}$, $Z \in \{X,Y\}$ and substituting it by the elements of the corresponding I_{Z_i}, ordered according with the ordenation of the $\underset{k}{H}$, $H \in \{S,W\}$.

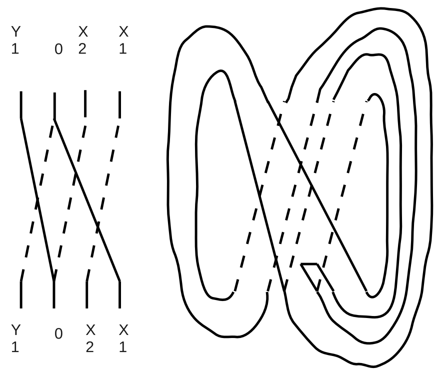

Figure 2: Construction of the sub-Lorenz template, $T_{(X,Y)}$ for $(X,Y) = (LRR0, RL0)$. Note that $X_0 = Y_0 = 0$.

3 Sub-Lorenz Templates

Now we will follow [16] to introduce sub-Lorenz templates and Williams zeta-functions.

We say that a Lorenz map f haves a double saddle connection if $f^n(0^-) = f^m(0^+) = 0$ for some n, m.

In this case the smallest intervals with extreme points $\{f^i(0^-), f^j(0^+) : 1 \leq i \leq n, 1 \leq j \leq m\}$, define a finite Markov partition for the semi-flow.

The restriction of the semiflow to this partition is called a Sub-Lorenz template.

Example 2 *For $(X,Y) = (LRR0, RL0)$, the sub-Lorenz template $T_{(X,Y)}$ is constructed in Figure 2:*

We identify each Sub-Lorenz template with the corresponding kneading invariant (X,Y) and denote it with $T_{(X,Y)}$. We associate to the template the transition matrix $A_{T_{(X,Y)}} = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } I_j \subset f(I_i) \\ 0 & \text{if } I_j \bigcap f(I_i) = \emptyset \end{cases}$$

Now we associate to $T_{(X,Y)}$ a labeled transition matrix $A_{T_{(X,Y}}(L,R) = [a'_{ij}]$, where

$$a'_{ij} = \begin{cases} L & \text{if } I_j \subset f(I_i) \text{ and } I_i \text{ is located on the left of } 0 \\ R & \text{if } I_j \subset f(I_i) \text{ and } I_i \text{ is located on the right of } 0 \\ 0 & \text{if } I_j \bigcap f(I_i) = \emptyset \end{cases}$$

In the previous example we have

$$A_{T(X,Y)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$A_{T(X,Y)}(L, R) = \begin{bmatrix} 0 & L & L \\ R & 0 & 0 \\ 0 & R & 0 \end{bmatrix}$$

Given a group F and a matrix A with entries in F, we define the *link-ring* $R(A)$ as follows:

1. For each sequence (called a cycle below)

$$i_1, i_2, \ldots, i_k, \;\; i_r \neq i_s \text{ for } r \neq s,$$

   such that the product

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} \neq 0,$$

   let $(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})$ be the equivalence class under cyclic permutations of this product. These equivalence classes are called ***free-knot symbols*** and the indices $i_1, i_2, \ldots, i_k$ are called *nodes* of the free knot symbol.

2. Define $R(A)$ to be the free abelian group generated by the free knot symbols defined in (1).

The nodes in the matrix correspond to the cells in the Markov partition. The cycles in the matrix correspond to periodic orbits on the Markov partition, and thus to periodic orbits in the flow. Since there is no natural way to choose a specific point for a periodic orbit, the natural invariant is the cyclic permutation class.

We do not permit products of letters to commute, since this usually corresponds to different orbits and consequently to different knot types, for example L^3R^2 and L^2RLR correspond, in the first case to the unknot and in the second to the trefoil knot, see Figure 3.

However, when we consider a product of words, it represents the union of two knots (i.e. **link**) and in this case we don't care which happens first, so **we permit products of free-knot symbols (words) to commute**. By a ***free-link symbol*** in the ring $R(A)$ is meant a product $x_1 \cdots x_l$ of free-knot symbols, no two of which have a node in common.

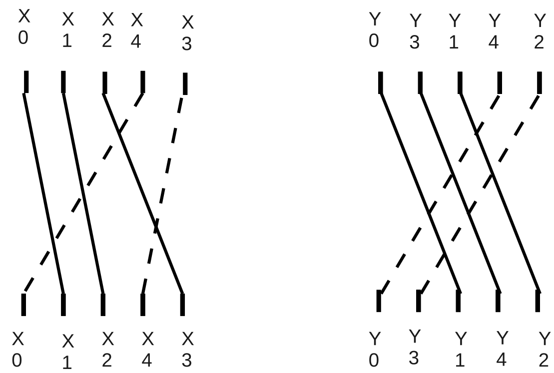

Figure 3: The knots corresponding to the sequences $X = (LLLRR)^\infty$ (the unknot) and $Y = (LLRLR)^\infty$ (the trefoil knot).

(Just as a product of two cycles can occur in a determinant, only if they have no node in common)

In [16], Williams defined the following determinant-like function and proved the following two results about it:

Definition 2 *Let $A_{T_{(X,Y)}}(L, R)$ be a labeled transition matrix of a sub-Lorenz template, then*

$$\textit{link-det}(I - A_{T_{(X,Y)}}(L, R)) = \sum_{free-link\ symbols} (-1)^l x_1 x_2 \cdots x_l.$$

Theorem 3 (Williams) *For Lorenz attractors $T_{(X,Y)}$ with a double saddle connection,*

$$\textit{link-det}(I - A_{T_{(X,Y)}}(L, R)) = \sum_{\mathcal{L}} (-1)^{|L|} fls(L),$$

where $\mathcal{L}$ is the collection of all links L in the attractor which have at most one point in each partition set, $|L|$ is the number of components in L, and $fls(L)$ means the free-link symbol of L.

Theorem 4 (Williams) *For Lorenz attractors $T_{(X,Y}$ with a double saddle connection, we have that*

$$exp\left(-\sum_{i=0}^{\infty} tr(A_{T_{(X,Y)}}(L, R)^i)/i\right) = \textit{link-det}(I - A_{T_{(X,Y)}}(L, R)).$$

From this result, we name $\zeta_W(T_{(X,Y)}) = \text{link-det}(I - A_{T_{(X,Y)}}(L,R))$, the ***Williams zeta function*** of the template $T_{(X,Y)}$.

Finally we state our factorization result about Williams zeta-functions:

Theorem 5 *Let $T_{(X,Y)*(S,W)}$ be a Sub-Lorenz template generated by a Renormalizable Lorenz map f, with a double saddle connection and kneading invariant $K(f) = (X,Y)*(S,W)$, then*

$$\zeta_W(T_{(X,Y)*(S,W)}) + 1 = \left[\zeta_W(T_{(X,Y)}) + 1\right] \times \left[(X,Y) * \zeta_W(T_{(S,W)}) + 1\right],$$

where $(X,Y) * \zeta_W(T_{(S,W)}) = \sum(-1)^l (X,Y) * x_1 \dots (X,Y) * x_l$ such that the sum is taken over all free-link symbols $x_1 \dots x_n$ of $T_{(S,W)}$.

Proof Denote $\mathcal{RB} = \bigcup\limits_{Z\in\{X,Y\}} \bigcup\limits_{0\leq i<|Z|} I_{Z_i}$.

i Let $\mathcal{P}_{(X,Y)*(S,W)}$ be the Markov partition associated to $(X,Y)*(S,W)$, and I be a cell of $\mathcal{P}_{(X,Y)*(S,W)}$, then $I \subset \mathcal{RB}$ or $I = J_k$ for some k such that $1 \leq k \leq |X| + |Y|$. Moreover, from (3) of remark 1 we have that $\sigma^n(\mathcal{RB}) \subset \mathcal{RB}$ for all n and, from (4), the free-knot symbols associated to nodes in $\mathcal{P}_{(X,Y)*(S,W)} \backslash \mathcal{RB}$ are exactly the same free-knot symbols from $T_{(X,Y)}$ and consequently the free-link symbols are also the same.

ii From (5) of Remark 1, the free-link symbols associated to knots in $\mathcal{RB}$ are exactly those that can be written in the form $(X,Y)*x_1 \cdots (X,Y)*x_l$, where $x_1 \cdots x_l$ are free-link symbols of $T_{(S,W)}$.

iii Finally, from (3) of Remark 1 there are no cycles, simultaneously with nodes in $\mathcal{RB}$ and in $\mathcal{P}_{(X,Y)*(S,W)} \backslash \mathcal{RB}$, so all free-link symbols of type $\alpha\beta$ where α is a free link symbol associated to nodes in $\mathcal{RB}$ and β is a free link symbol associated to nodes in $\mathcal{P}\backslash\mathcal{RB}$ will be terms of the zeta-function of $T_{(X,Y)*(S,W)}$.

Finally, from (i), (ii) and (iii) we have that

$$\zeta_W(T_{(X,Y)*(S,W)}) = \zeta_W(T_{(X,Y)}) + (X,Y)*\zeta_W(T_{(S,W)}) + (X,Y) \times (X,Y)*\zeta_W(T_{(S,W)}).$$

$\square$

4 Twist-zeta function

In this section we will present a factorization formula for the twist-zeta function presented by Sullivan in [14].

A ribbon is an embedded annulus or Möbius band in S^3. Like knots and templates, ribbons can be braided. A ribbon which has a braid presentation such that each crossing of one strand over another is positive and each twist in each strand is positive, will be called a positive ribbon. The core and boundary of positive ribbons are positive braids.

Definition 3 *If R is a ribbon and $b(R)$ is a braid presentation of R, we define the computed twist*

$$\tau_c(R) = 2n + t,$$

where t is the sum of the half twists in the strands of $b(R)$ and n is the number of strands of the core.

In [14], Sullivan proved that τ_c is an isotopy invariant of positive ribbons over positive braid presentations, so the definition is consistent.

Definition 4 *Given a template, T and an orbit O on T, we define the ribbon $R(T, O)$, as the ribbon defined by the unit normal bundle of O.*

Sullivan proved that, for positive templates, the number of closed orbits with a given computed twist is finite. This permited him to formulate the following definition:

Definition 5 *For a given template T, let $T_{q'}$ be the number of closed orbits with computed twist q'. Let $\mathcal{T}_q = \sum_{q'|q} q' T_{q'}$. Define the Sullivan zeta function of the template to be the exponential of a formal power series:*

$$\zeta_T^S(t) = \exp\left(\sum_{q=2}^{\infty} \mathcal{T}_q \frac{t^q}{q}\right).$$

We now define a twist matrix $A(t) = [a_{ij}]$ whose entries are non-negative powers of t and 0's, by considering the contribution to τ_c as an orbit goes from one element of a Markov partition to another. Let $a_{ij} = 0$ if there is no branch going from the i-th to the j-th partition element and $a_{ij} = t^{q_{ij}}$ if there is such a branch, where q_{ij} is the amount of computed twist an orbit picks up as it travels from the i-th to the j-th partition element.

In the case of a sub-Lorenz template $T_{(X,Y)}$, since we have one curl in each ribbon from one partition element to other, see Figure 2 then we obtain $A_{T(X,Y)}(t)$ simply substituting each 1 element of the transition matrix by t^2, i.e.

$$A_{T(X,Y)}(t) = A_{T(X,Y)}(t^2, t^2).$$

Sullivan proved the following theorem in [14].

Theorem 6 *For any template T and any allowed choice of $A(t)$ we have*

$$\zeta_T^S(t) = \frac{1}{det(I - A(t))}.$$

For a sub-Lorenz template $T_{(X,Y)}$, denote

$$\zeta_{T_{(X,Y)}}^S(L,R) = \frac{1}{det(I - A_{T_{(X,Y)}}(L,R))}.$$

Following this notation, we have that $\zeta_{T_{(X,Y)}}^S(t) = \zeta_{T_{(X,Y)}}^S(t^2,t^2)$.

We will now state our factorization result.

Theorem 7 *For a reducible kneading pair $(X,Y) * (S,W)$ with both (X,Y) and (S,W), admissible finite Lorenz pairs we have that*

$$\zeta_{T_{(X,Y)*(S,W)}}^S(t^2,t^2) = \zeta_{T_{(X,Y)}}^S(t^2,t^2)\zeta_{T_{(S,W)}}^S(t^{2|X|},t^{2|Y|}).$$

Proof From Remark 1, the Markov partition $\mathcal{P}$ associated to $T_{(X,Y)*(S,W)}$ can be splited as $\mathcal{P} = \mathcal{RB} \cup \mathcal{P} \backslash \mathcal{RB}$ in such a way that all the iterates of each periodic orbit are exclusively contained in $\mathcal{RB}$ or in $\mathcal{P}\backslash\mathcal{RB}$. So we can split the sum

$$\sum_{q=2}^{\infty} \mathcal{T}_q \frac{t^q}{q} = \sum_{q=2}^{\infty} \mathcal{T}_q\left(\mathcal{RB}\right)\frac{t^q}{q} + \sum_{q=2}^{\infty} \mathcal{T}_q\left(\mathcal{P}\backslash\mathcal{RB}\right)\frac{t^q}{q}$$

where $\mathcal{T}_q\left(\mathcal{RB}\right)$ (respectively $\mathcal{T}_q\left(\mathcal{P}\backslash\mathcal{RB}\right)$) means simply that we are counting orbits contained in $\mathcal{RB}$ (respectively in $\mathcal{P}\backslash\mathcal{RB}$).

From (4) of Remark 1, $\exp\left(\sum_{q=2}^{\infty} \mathcal{T}_q\left(\mathcal{P}\backslash\mathcal{RB}\right)\frac{t^q}{q}\right) = \zeta_{T_{(X,Y)}}^S(t^2,t^2)$. On the other hand, from (5) of Remark 1, each ribbon leaving a cell I_k makes $|X|$ curls without splitting if I_k is on the left of 0 and makes $|Y|$ curls without splitting if I_k is on the left of 0, before reenter in $\mathcal{BL}$.

Moreover $\sigma^{|X|}(I_k) \cap I_{k'}$ (respectively $\sigma^{|Y|}(I_k) \cap I_{k'}$) is not empty if and only if $\sigma(P_k) \cap \sigma(P_{k'}) \neq \emptyset$, so $\exp\left(\sum_{q=2}^{\infty} \mathcal{T}_q\left(\mathcal{RB}\right)\frac{t^q}{q}\right) = \zeta^S_{T_{(S,W)}}(t^{2|X|}, t^{2|Y|})$ and the result follows.

$\square$

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