The Classification of the Epimorphisms from the Artin-Tits groups of type $A(\widetilde{A}_n)$ to $W(\widetilde{A}_n)$

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Abstract

We present a complete classification for the epimorphisms from the affine Artin-Tits groups $A(\widetilde{A}_n)$ to the corresponding Coxeter groups $W(\widetilde{A}_n)$, for $n \geq 1$.

Keywords: Affine Artin-Tits groups, Coxeter groups, Classification problem

1. Introdution

Artin-Tits groups are a widely studied class of groups. We will study a particular sub-family of this groups the $A(\widetilde{A}_n)$ type Artin-Tits groups. In order to do so we start by some basic definitions in order to be able to explain our main goal.

Define a *Coxeter matrix* of rank n as a $n \times n$ symmetric matrix, $M = (m_{i,j})$, that verifies: $m_{i,i} = 1$ for all $i = 1, \ldots, n$, and $m_{i,j} \in \{2, 3, \ldots\} \cup \{\infty\}$, for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. Let M be a Coxeter matrix. The *Coxeter graph* associated to M is the labelled graph, Γ, defined as follows. The set of vertices of Γ is $\{1, \ldots, n\}$. If $m_{i,j} = 2$ then there is no edge between i and j, if $m_{i,j} = 3$, then there is a non-labelled edge between i and j, and, finally, if $m_{i,j} > 3$ or $m_{i,j} = \infty$, then there is an edge between i and j labelled by $m_{i,j}$.

Let G be a group. For $a, b \in G$ and $n \in \mathbb{N}$ we define the word

$$prod_n(a,b) \begin{cases} (ab)^{\frac{n}{2}} & \text{if } n \equiv 0 (\text{mod } 2) \\ (ab)^{\frac{n-1}{2}} a & \text{if } n \equiv 1 (\text{mod } 2) \end{cases}$$

Define the *Coxeter group* W associated to the Coxeter graph Γ as the group presented by

$$W = W(\Gamma) = \left\langle \; s_1, s_2, \ldots, s_n \;\middle|\; \begin{array}{l} s_i^2 = 1, \quad i \in \{1, \ldots, n\} \\ prod_{m_{i,j}}(s_i, s_j) = prod_{m_{j,i}}(s_j, s_i) \\ \qquad\qquad i \neq j \text{ and } m_{i,j} \neq \infty \end{array} \right\rangle,$$

where $M = (m_{i,j})$ is the Coxeter matrix of Γ. Define the *Artin-Tits group* associated to Γ to be the group presented by

$$A = A(\Gamma) = \left\langle \; s_1, s_2, \ldots, s_n \;\middle|\; \begin{array}{l} prod_{m_{i,j}}(s_i, s_j) = prod_{m_{j,i}}(s_j, s_i) \\ \qquad\qquad i \neq j \text{ and } m_{i,j} \neq \infty \end{array} \right\rangle.$$

[1]Author partially supported by FCT project PTDC/MAT/73544/2006

Figure 1: Coxeter graph A_{n-1}

If the group $W(\Gamma)$ is finite, then we say that $A(\Gamma)$ is an *Artin-Tits group of spherical type*.

The Artin-Tits group of type of spherical type $A(A_{n-1})$ is defined by the Coxeter graph in figure 1. These groups are also known as Artin braid groups. Let us recall the Artin presentation (see [A1]) of the braid group $A(A_{n-1})$, for $n > 2$:

$$A(A_{n-1}) = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geq 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (i = 1, \ldots, n-2) \end{array} \right\rangle.$$

Recall (see [B]) that the Garside element of $A(A_{n-1})$ is $\Delta = \sigma_1 \cdots \sigma_{n-1}\sigma_1 \cdots \sigma_{n-2} \cdots \sigma_1\sigma_2\sigma_1$. A positive braid is a braid which can be written only with positive powers of the generators. A simple braid is a positive braid which divides Δ.

The affine group $A(\widetilde{A}_{n-1})$, for $n > 2$, is the defined by the Coxeter graph in figure 2.

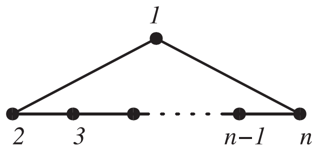

Figure 2: Coxeter graph $\tilde{A}_{n-1}$

This is not a spherical type Artin-Tits group. We can obtain a presentation of the Afine Artin-Tits group $A(\widetilde{A}_{n-1})$, we just add to the Artin presentation $A(A_{n-1})$ a generator σ_n and the relations:

$$\sigma_n\sigma_1\sigma_n = \sigma_1\sigma_n\sigma_1$$

$$\sigma_n\sigma_{n-1}\sigma_n = \sigma_{n-1}\sigma_n\sigma_{n-1}$$

$$\sigma_n\sigma_i = \sigma_i\sigma_n \text{ for } i = 2, \ldots, n-2$$

The group $A(\widetilde{A}_1)$ is defined by the Coxeter graph in figure 3. This is a free group on two generators and we will deal with it separately in section 9.

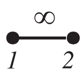

Figure 3: Coxeter graph $\tilde{A}_1$

For a given Coxeter graph Γ, associated to a $n \times n$ Coxeter matrix, there is an natural epimorphism $\mu : A(\Gamma) \to W(\Gamma)$ defined by $\mu(s_i) = s_i$ for $i = 1, \ldots, n$.

Main goal: We will be interested in all the other epimorphisms in the case of the affine group of type $A(\widetilde{A}_{n-1})$. We will present a complete classification of these epimorphisms.

A full classification of the epimorphisms from the spherical type Artin-Tits groups into the corresponding Coxeter group can be found in [A]. This is the base point for several other results (see for example [CP]) concerning the classification of epimorphism of the Artin-Tits type groups to the Coxeter groups associated. We will start by extending the result in [A] to the affine group $A(\widetilde{A}_{n-1})$, $n > 2$. In section 3, we start to reduce the number of candidates to be an epimorphism by using the relations in $A(\widetilde{A}_{n-1})$, $n > 2$. In section 4, we will use another criterium on $\mathbb{Z}^n$, $n > 2$ to reduce even more our list. In section 5, we present a characterization of the kernel of a specific morphism. We proceed in section 6 by computing, under automorphisms of $W(\widetilde{A}_{n-1})$, the images of a fixed type of elements in $W(\widetilde{A}_{n-1})$, $n > 2$. Using the characterization obtained in section 5 and the results in section 6, we write, in section 7, a complete list of epimorphisms from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$, $n > 2$. We compute the classes of epimorphisms, up to automorphisms of $W(\widetilde{A}_{n-1})$, $n > 2$, in section 8. We present the special case of $A(\widetilde{A}_1)$ in section 9. In section 10 we present our main result, a complete classification for the epimorphisms from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$, $n > 1$.

2. Epimorphisms from $A(\widetilde{A}_{n-1})$ to de symmetric group S_n.

We will present, in this section, the list of epimorphisms from $A(\widetilde{A}_{n-1})$ to de symmetric group S_n, $n > 2$. To do so we start by recalling the result presented in [A] and after it we will see as it proof can be adapted to obtain our list epimorphisms from $A(\widetilde{A}_{n-1})$ to S_n.

Theorem 2.1. *[A] The possible representations of $A(A_{n-1})$ in the symetric group S_n are:*

1. $\sigma_i = (i, i+1)$.
2. $\sigma_1 = (1,2)(3,4)(5,6)$, $D = (1,2,3)(4,5)$ *for* $n = 6$.
3. $\sigma_1 = (1,2,3,4)$, $D = (1,2)$ *for* $n = 4$.
4. $\sigma_1 = (1,3,2,4)$, $D = (1,2,3,4)$ *for* $n = 4$.

We present now our result, which is a direct consequence of Artin's result:

Theorem 2.2. *The possible representations of $A(\widetilde{A}_{n-1})$ in the symetric group S_n are:*

1. $\sigma_i = (i, i+1)$ *and* $\sigma_n = (1, n)$.
2. $\sigma_1 = (1,2)(3,4)(5,6)$, $D = (1,2,3)(4,5)$ *and* $\sigma_6 = (1,5)(2,3)(4,6)$ *for* $n = 6$.
3. $\sigma_1 = (1,2,3,4)$, $D = (1,2)$ *and* $\sigma_4 = (1,2,4,3)$ *for* $n = 4$.
4. $\sigma_1 = (1,2,3,4)$, $D = (1,2)$ *and* $\sigma_4 = (1,3,4,2)$ *for* $n = 4$.
5. $\sigma_i = (i, i+1)$ *and* $\sigma_4 = \sigma_2$, *for* $n = 4$.
6. $\sigma_1 = (1,3,2,4)$, $D = (1,2,3,4)$ *and* $\sigma_4 = (1,3,4,2)$ *for* $n = 4$.
7. $\sigma_1 = (1,3,2,4)$, $D = (1,2,3,4)$ *and* $\sigma_4 = (1,2,4,3)$ *for* $n = 4$.

Proof. The result follows from Artin's proof by slightly changing the preliminary lemmas. This is just done by assuming the existence of an extra generator and the relations containing it.

As in Artin's paper we define for $k > i$:

$$D_{ki} = \sigma_k \cdots \sigma_n \sigma_1 \cdots \sigma_{i-1}.$$

and we also have as in (6):

$$D_{ki}^{n-k+i+1} = (\sigma_k D_{ki})^{n-k+i}.$$

Regarding lemma 1 from [A], we must add or if σ_1 is commutative with σ_n. The proof remains unchanged.

Lemma 2, 4 and 6 will be use has they are, without changes.

As to Lemma 3 we must add the case $k > i$ and then we must replace the integers t by $n - t$. The proof is obtained by rewriting, the original proof, with a shifting of the indices.

The conclusion of Lemma 5 is the same and in the proof we just have to put an upper limit on $i \geq 3$ and write it $n > i \geq 3$.

Now the proof of the theorem uses the adaptation of Artin's lemmas in the same way. We just have to search, among all possible permutations, for the images of the $n-$th generator for the $n = 4, 6$ special cases.

□

3. Reducing the possible epimorphisms

The Coxeter group $W(\widetilde{A}_{n-1})$ is isomorphic to $\mathbb{Z}^{n-1} \rtimes S_n$ (see [Bou]). For better computations we use the following semidirect product $\mathbb{Z}^n \rtimes S_n$, where S_n acts on $\mathbb{Z}^n$ by permuting its coordinates but we will impose that the sum of all coordinates is 0. Typically an element in $W(\widetilde{A}_{n-1})$ is represented by $[x_1, \ldots, x_n]s$, where $x_i \in \mathbb{Z}, \sum x_i = 0$ and $s \in S_n$.

We will assume that:

$$\xi(\sigma_i) = [u_{i1}, \ldots, u_{in}]\xi_l(\sigma_i),$$

where ξ_l corresponds to the epimorphism in the l^{th} case in theorem 2.2.

3.1. Case $l = 1$

Proposition 3.1. *Let σ_i be the generators of $A(\widetilde{A}_{n-1})$. We define ξ from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$ as:*

$$\xi(\sigma_i) = [\underbrace{y, \ldots, y}_{i-1}, -(n-2)y - x_i, x_i, y, \ldots, y](i, i+1), n > i \geq 1$$

$$\xi(\sigma_n) = [x_n, y, \ldots, y, -(n-2)y - x_n](1, n).$$

For all $x_i \in \mathbb{Z}$, $\xi(\sigma_i)$ verify the braid relations.

Proof. We will start by verifying the type 3 relations:

1. Case $(i, i+1)$ with $i + 1 \leq n$ and $i \geq 1$.
   We have
   $\xi(\sigma_i)\xi(\sigma_{i+1})\xi(\sigma_i) =$
   $= [y, \ldots, y, -(n-2)y - x_i, x_i, y, \ldots, y](i, i+1)[y, \ldots, y, -(n-2)y - x_{i+1}, x_{i+1}, y, \ldots, y](i+1, i+2)\xi(\sigma_i) =$
   $= [2y, \ldots, 2y, -(n-2)y - x_i + y, y - (n-2)y - x_{i+1}, x_i + x_{i+1}, 2y, \ldots, 2y](i, i+2, i+1)[y, \ldots, y, -(n-2)y - x_i, x_i, y, \ldots, y](i, i+1) =$
   $= [3y, \ldots, 3y, -2(n-2)y - x_i - x_{i+1} + y, y - (n-2)y, y + x_i + x_{i+1}, 3y, \ldots, 3y](i, i+2).$
   and
   $\xi(\sigma_{i+1})\xi(\sigma_i)\xi(\sigma_{i+1}) =$
   $= [y, \ldots, y, -(n-2)y - x_{i+1}, x_{i+1}, y, \ldots, y](i+1, i+2)[y, \ldots, y, -(n-2)y - x_i, x_i, y, \ldots, y](i, i+1)\xi(\sigma_{i+1}) =$
   $= [2y, \ldots, 2y, -2(n-2)y - x_i - x_{i+1}, y + x_i, x_{i+1} + y, 2y, \ldots, 2y](i, i+1, i+2)[y, \ldots, y, -(n-2)y - x_{i+1}, x_{i+1}, y, \ldots, y](i+1, i+2) =$
   $= \xi(\sigma_i)\xi(\sigma_{i+1})\xi(\sigma_i)$

2. Case $(1,n)$.
We have
$\xi(\sigma_1)\xi(\sigma_n)\xi(\sigma_1) = [-(n-2)y-x_1, x_1, y, \dots, y](1,2)[x_n, y, \dots, y, -(n-2)y-x_n](1,n)\xi(\sigma_1) =$
$= [-(n-2)y - x_1 + y, x_1 + x_n, 2y, \dots, 2y, y - (n-2)y - x_n](1,n,2)[-(n-2)y - x_1, x_1, y, \dots, y](1,2) = [-(n-2)y+y, x_1+x_n+y, 3y, \dots, 3y, y-2(n-2)y-x_n-x_1](2,n)$
and
$\xi(\sigma_n)\xi(\sigma_1)\xi(\sigma_n) = [x_n, y, \dots, y, -(n-2)y-x_n](1,n)[-(n-2)y-x_1, x_1, y, \dots, y](1,2)\xi(\sigma_n) =$
$= [x_n+y, y+x_1, 2y, \dots, 2y, -2(n-2)y-x_1-x_n](1,2,n)[x_n, y, \dots, y, -(n-2)y-x_n](1,n) =$
$\xi(\sigma_1)\xi(\sigma_n)\xi(\sigma_1)$

Verify type 2 relations is much easier because it suffices to notice that: in each product of two generators the permutations will only act on coordinates that are equal (to y).

□

Proposition 3.2. *It does not exist any other solution to the braid equations.*

Proof. Suppose that we have another solution for the braid equations. Let us show first that:

$$u_{ik} = u_{i(k+1)} \text{ for } k \neq i, i+1.$$

We just have to see that from the relation

$$\sigma_k\sigma_i = \sigma_i\sigma_k$$

we obtain, among others, the equation

$$u_{kk} + u_{ik} = u_{kk} + u_{i(k+1)}.$$

We can conclude that $u_{ik} = u_{ij}$ where $k, j \neq i, i+1$.
Now we must show that:

$$u_{ik} = u_{(i+1)j} \text{ for } k \neq i, i+1 \text{ and } j \neq i+1, i+2.$$

For this we will use the type 3 braid relation

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

from which we get the equation

$$2u_{ik} + u_{(i+1)k} = u_{ik} + 2u_{(i+1)k}.$$

So the new solution can differ, from the previous ones, for each generator σ_i only in coordinates $i, i+1$ if $i < n$ or $1, n$ if $i = n$. But we have only one coordinate to change because the equations

$$\sum_j u_{ij} = 0$$

fix the second one. We obtain exactly previous solutions.

□

3.2. Case $l = 2$

In this case we are in $A(\widetilde{A}_5)$, and we obtain, from the braid equations, the following solutions for ξ:

$\xi(\sigma_1) = [x_1 - x_2 + x_5 - x_4, -x_1 + x_2 - x_5 + x_4, -x_1 + x_3, x_1 - x_3, -x_2, x_2](1,2)(3,4)(5,6)$
$\xi(\sigma_2) = [x_1 - x_2 + x_5 - x_3 - x_4, -x_5 - x_1 + x_2, -x_1 + x_2 - x_5 + x_3 + x_4, -x_5, x_5 + x_1 - x_2, x_5](1,3)(2,5)(4,6)$,
$\xi(\sigma_3) = [-x_2 + x_5 - x_4, -x_3, x_3, x_2 - x_5 + x_4, -x_2, x_2](1,4)(2,3)(5,6)$,
$\xi(\sigma_4) = [x_1 - x_2 + x_5 - x_4, -x_1 + x_2 - x_5 + x_4, -x_1 + x_2 - x_5 + x_3, -x_5, x_1 - x_2 + x_5 - x_3, x_5](1,2)(3,5)(4,6)$,
$\xi(\sigma_5) = [x_1 - x_2 + x_5 - x_3 - x_4, -x_1, -x_1 + x_2 - x_5 + x_3 + x_4, x_1, -x_2, x_2](1,3)(2,4)(5,6)$,
$\xi(\sigma_6) = [-x_4, -x_3, x_3, -x_5, x_4, x_5](1,5)(2,3)(4,6)$
with $x_i \in \mathbb{Z}$.

3.3. Case $l = 3$

In this case we are in $A(\widetilde{A}_3)$, and we obtain, from the braid equations, the following solutions for ξ:

$\xi(\sigma_1) = [x_3, -x_3 - x_1 - x_2, x_1, x_2](1,2,3,4)$,
$\xi(\sigma_2) = [-x_2 - x_1, -x_3, x_1, x_2 + x_3](1,3,4,2)$,
$\xi(\sigma_3) = [x_3, -x_3 - x_1 - x_2, x_1, x_2](1,2,3,4)$,
$\xi(\sigma_4) = [x_3, -x_2 - x_3, x_2 + x_1, -x_1](1,2,4,3)$
with $x_i \in \mathbb{Z}$.

3.4. Case $l = 4$

In this case we are in $A(\widetilde{A}_3)$, and we obtain, from the braid equations, the following solutions for ξ:

$\xi(\sigma_1) = [-x_1, -x_2 - x_3, x_2, x_1 + x_3](1,2,3,4)$,
$\xi(\sigma_2) = [-x_3 - x_1 - x_2, x_1, x_2, x_3](1,3,4,2)$,
$\xi(\sigma_3) = [-x_1, -x_2 - x_3, x_2, x_1 + x_3](1,2,3,4)$,
$\xi(\sigma_4) = [-x_3 - x_1 - x_2, x_1, x_2, x_3](1,3,4,2)$
with $x_i \in \mathbb{Z}$.

3.5. Case $l = 5$

In this case we are in $A(\widetilde{A}_3)$, and we obtain, from the braid equations, the following solutions for ξ:

$\xi(\sigma_1) = [x_1, -x_1 - 2x_3, x_3, x_3](1,2)$,
$\xi(\sigma_2) = [x_3, x_4, -2x_3 - x_4, x_3](2,3)$,
$\xi(\sigma_3) = [x_3, x_3, x_2, -2x_3 - x_2](3,4)$,
$\xi(\sigma_4) = [x_3, x_4, -2x_3 - x_4, x_3](2,3)$
with $x_i \in \mathbb{Z}$.

3.6. Case $l = 6$

In this case we are in $A(\widetilde{A}_3)$, and we obtain, from the braid equations, the following solutions for ξ:

$\xi(\sigma_1) = [x_2, x_3, x_1, -x_2 - x_3 - x_1](1,3,2,4)$,
$\xi(\sigma_2) = [x_2, -x_2 - x_1, x_1 + x_3, -x_3](1,3,4,2)$,
$\xi(\sigma_3) = [x_2 + x_3 + x_1, -x_1, -x_2, -x_3](1,4,2,3)$,
$\xi(\sigma_4) = [x_2, -x_2 - x_1, x_1 + x_3, -x_3](1,3,4,2)$
with $x_i \in \mathbb{Z}$.

3.7. Case $l = 7$

In this case we are in $A(\widetilde{A}_3)$, and we obtain, from the braid equations, the following solutions for ξ:

$\xi(\sigma_1) = [x_2 + x_3 + x_1, -x_1, -x_3, -x_2](1,3,2,4)$,
$\xi(\sigma_2) = [x_2 + x_3 + x_1, -x_1 - x_2, -x_1 - x_3, x_1](1,3,4,2)$,
$\xi(\sigma_3) = [x_2, x_3, -x_2 - x_3 - x_1, x_1](1,4,2,3)$,
$\xi(\sigma_4) = [x_1 + x_2, -x_1, -x_2 - x_3 - x_1, x_1 + x_3](1,2,4,3)$
with $x_i \in \mathbb{Z}$.

4. One criterium to be an Epimorphism

We will present sufficient condition for a morphism, from a group G to $\mathbb{Z}^n$, to be an epimorphism (for details see for [RG-S]).

Let us introduce first some notation: Let $\zeta : G \to \mathbb{Z}^n$ be a morphism and $g_1, \dots, g_k$ a generating system of G. Denote by M the $k \times n$ matrix formed by $\zeta(g_1), \dots, \zeta(g_k)$,

$$det_{(i_1,\dots,i_p)} = det \begin{pmatrix} \zeta(g_{i_1})_1 & \cdots & \zeta(g_{i_1})_p \\ \vdots & & \vdots \\ \zeta(g_{i_p})_1 & \cdots & \zeta(g_{i_p})_p \end{pmatrix}$$

and

$$det'_{(i_1,\dots,i_p)} = det \begin{pmatrix} \zeta(g_{i_1})_p & \cdots & \zeta(g_{i_1})_n \\ \vdots & & \vdots \\ \zeta(g_{i_p})_p & \cdots & \zeta(g_{i_p})_n \end{pmatrix}$$

Lemma 4.1. *In the previous conditions, if $gcd(det_{i_1,\dots,i_n}) = 1$ then $gcd(det_{i_1,\dots,i_r}) = 1$ (resp. $gcd(det'_{i_1,\dots,i_r}) = 1$) for $r = 1, \dots, n$.*

Proof. It is sufficient to notice that, for each r $det_{i_1,\dots,i_n}$ is a linear combinations of all $det_{i_1,\dots,i_r}$ (resp. $det'_{i_1,\dots,i_r}$) and if $gcd(det_{i_1,\dots,i_r}) \neq 1$ then we have a contradiction.

□

Remark that the previous lemma implies that the *gcd* of all elements in columns of the matrix are 1.

Proposition 4.2. *Let $\zeta : G \to \mathbb{Z}^n$ be a morphism and $g_1, \dots, g_k$ a generating system of G. Then $gcd(det_{i_1,\dots,i_n}) = 1$ for some $\{i_1, \dots, i_n\} \subset \{1, \dots, k\}$ with $\#\{i_1, \dots, i_n\} = n$ if and only if ζ is an epimorphism.*

5. Characterizing a kernel

Let $n > 2$. Consider de following sequence:

$$A(\widetilde{A}_{n-1}) \xrightarrow{\xi} W(\widetilde{A}_{n-1}) \simeq \mathbb{Z}^{n-1} \rtimes S_n \xrightarrow{p} S_n$$

We will start this section by computing $\xi(ker(p \circ \xi))$.

Consider de Cayley graph $\mathcal{G}$ of $W(\widetilde{A}_{n-1})$ and its projection $p(\mathcal{G})$. Let us denote by $Ch(p(\mathcal{G}))$ the set of paths in $p(\mathcal{G})$ with origin in id. Let $\gamma \in Ch(p(\mathcal{G}))$ be a path described by de sequence

$(l_1, \ldots, l_k)$ the labels of the edges in γ. Note that some of the l_i can be negative going a long an edge in the inverse sense. We define a morphism ϕ from $Ch(p(\mathcal{G}))$ to $A(\widetilde{A}_{n-1})$ by:

$$\phi(\gamma) = \sigma_{|l_1|}^{sg(l_1)} \cdots \sigma_{|l_k|}^{sg(l_k)}.$$

Proposition 5.1. *In the previous conditions:*

$$\phi(\pi_1(p(\mathcal{G}), id)) = ker(p \circ \xi).$$

We will construct a particular generating set, which can be divided in two parts. One appears naturally from the fact that $A(A_{n-1})$ injects naturally in $A(\widetilde{A}_{n-1})$. These generators are the generators of the pure braid group $P(A(A_{n-1}))$, the kernel of the standard epimorphism μ. This also allow us to build a maximal tree of $p(\mathcal{G})$ based on id which do not contain any edge labelled n and no branches leaving id with length greater than $\frac{n(n-1)}{2}$. The second one is characterized by the following lemma:

Lemma 5.2. *All generators, g, of $\phi(\pi_1(p(\mathcal{G}), id))$ that include in its writting the n^{th} generator can be written as:*

$$g = s\sigma_n t^{-1}.$$

For some simple elements $s, t \in A_{n-1}$.

Proof. It suffices to notice that all the vertices of $p(\mathcal{G})$ are the simple elements in A_{n-1}. We also have that a maximal tree in the Cayley graph of S_n is a maximal tree of $p(\mathcal{G})$. So the only way to obtain a generator with the n^{th} generator is by going from the identity to some vertex s then along the edge labeled n to the vertex t and return to the identity.

□

6. The behavior of $W(\widetilde{A}_{n-1})$ automorphisms

In [F] the author proves that the automorphisms of $W(\widetilde{A}_{n-1})$, $n > 1$, are all inner by graph. This means that all automorphisms are inner automorphisms (conjugations) or a Coxeter graph automorphism. Using this result we will eliminate some of the parameters. We start by using the inner automorphisms.

Proposition 6.1. *Let ξ from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$ be the morphism defined in proposition 3.1 and $n > 2$. Then, modulo inner automorphisms of $W(\widetilde{A}_{n-1})$, we assume that x_i, for $i = 1, \ldots, n-1$, can appear only in the expression defining $\xi(\sigma_n)$.*

Proof. Consider de element $w_{i,t} = [\underbrace{0, \cdots, 0}_{i-1}, t, 0, \cdots, 0, -t] \in W(\widetilde{A}_{n-1})$. Let $\psi_{w_{i,t}}$ be the inner automorphism of $W(\widetilde{A}_{n-1})$ associated to $w_{i,t}$. So we have

$$\psi_{w_{i+1,t}}(\xi(\sigma_i)) = [\underbrace{y, \ldots, y}_{i-1}, -(n-2)y - x_i - t, x_i + t, y, \ldots, y](i, i+1);$$

$$\psi_{w_{i+1,t}}(\xi(\sigma_{i+1})) = [\underbrace{y, \ldots, y}_{i}, -(n-2)y - x_{i+1} + t, x_{i+1} - t, y, \ldots, y](i+1, i+2);$$

$$\psi_{w_{i+1,t}}(\xi(\sigma_n)) = [x_n + t, y, \dots, y, -(n-2)y - x_n - t](1, n);$$

$$\psi_{w_{i+1,t}}(\xi(\sigma_j)) = \xi(\sigma_j) \text{ for } j \neq i, i+1, n.$$

So we can put all x_i only in the image of $\xi(\sigma_n)$, with $i = 1, \cdots, n-1$, conjugating recursively by the elements

$$w_{2,-x_1}, w_{3,-x_1-x_2}, \cdots, w_{n-1,\sum_{i=1,\cdots,n-2} -x_i} \text{ and } w_{1,\sum_{i=1,\cdots,n-1} x_i}$$

So for the first conjugation:

$$\psi_{w_{2,-x_1}}(\xi(\sigma_1)) = [-(n-2)y, 0, y, \dots, y](1, 2);$$

$$\psi_{w_{2,-x_1}}(\xi(\sigma_2)) = [y, -(n-2)y - x_2 - x_1, x_2 + x_1, y, \dots, y](2, 3);$$

$$\psi_{w_{2,-x_1}}(\xi(\sigma_n)) = [x_n - x_1, y, \dots, y, -(n-2)y - x_n + x_1](1, n);$$

$$\psi_{w_{2,-x_1}}(\xi(\sigma_j)) = \xi(\sigma_j) \text{ for } j \neq 1, 2, n.$$

The second conjugation acts as follows:

$$\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_1)) = \psi_{w_{3,-x_1-x_2}}([-(n-2)y, 0, y, \dots, y](1, 2)) = \psi_{w_{2,-x_1}}(\xi(\sigma_1));$$

$$\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_2)) = \psi_{w_{3,-x_1-x_2}}([y, -(n-2)y - x_2 - x_1, x_2 + x_1, y, \dots, y](2, 3)) =$$

$$= [y, -(n-2)y, 0, y, \dots, y](2, 3);$$

$$\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_3)) = \psi_{w_{3,-x_1-x_2}}(\xi(\sigma_3)) =$$

$$= \psi_{w_{3,-x_1-x_2}}([y, y, -(n-2)y - x_3, x_3, y, \dots, y](3, 4)) =$$

$$= [y, y, -(n-2)y - x_3 - x_2 - x_1, x_3 + x_2 + x_1, y, \dots, y](3, 4);$$

$$\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_n)) = [x_n - 2x_1 - x_2, y, \dots, y, -(n-2)y - x_n + 2x_1 + x_2](1, n);$$

$$\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_j)) = \xi(\sigma_j) \text{ for } j \neq 1, 2, 3, n.$$

and so on.

The case of the last inner automorphism is slightly different:

$$\psi_{w_{1,\sum_{i=1,\cdots,n-1} x_i}}\psi_{w_{n-1,\sum_{i=1,\cdots,n-2} -x_i}} \cdots \psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_{n-1})) =$$

$$= \psi_{w_{1,\sum_{i=1,\cdots,n-1} x_i}}([y,\ldots,y,-(n-2)y-x_{n-1}-x_{n-2}-\cdots-x_1,x_{n-1}+x_{n-2}+\cdots+x_1](n-1,n)) =$$

$$= [x_{n-1}+x_{n-2}+\cdots+x_1,y,\ldots,y,-x_{n-1}-x_{n-2}-\cdots-x_1]$$

$$[y,\ldots,y,-(n-2)y-x_{n-1}-x_{n-2}-\cdots-x_1,x_{n-1}+x_{n-2}+\cdots+x_1](n-1,n)$$

$$[-x_{n-1}-x_{n-2}-\cdots-x_1,y,\ldots,y,x_{n-1}+x_{n-2}+\cdots+x_1] =$$

$$= [y,\ldots,y,-(n-2)y,0](n-1,n).$$

In the end of this process we have all the x_i appear only in the expression defining $\xi(\sigma_n)$.

□

Proposition 6.2. *Let ξ from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$ be the morphism defined in proposition 3.1 and $n > 2$. Let $y,p \in \mathbb{Z}$, $\xi_{y,p}$ from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$ be the morphism defined by*

$$\xi_{y,p}(\sigma_i) = [\underbrace{y,\ldots,y}_{i-1},-(n-2)y,0,y,\ldots,y](i,i+1);$$

$$\xi_{y,p}(\sigma_n) = [p,y,\ldots,y,-(n-2)y-p](1,n);$$

Then, for $p = x_n-(n-5)x_1-(n-6)x_2-\cdots-x_{n-4}+2x_{n-1}+x_{n-2}$, $\xi_{y,p}$ and ξ are equal up to automorphisms of $W(\widetilde{A}_{n-1})$.

Proof. After conjugating ξ by $\psi = \psi_{w_{1,\sum_{i=1,\cdots,n-1} x_i}}\psi_{w_{n-1,\sum_{i=1,\cdots,n-2} -x_i}}\cdots\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}$, we obtain:

$$\psi(\xi(\sigma_i)) = [\underbrace{y,\ldots,y}_{i-1},-(n-2)y,0,y,\ldots,y](i,i+1) = \xi_{y,p}(\sigma_i), \text{ for } 1 \le i < n.$$

To see what happens to $\psi(\xi(\sigma_n))$ we will compute the last conjugation:

$$\psi_{w_{1,\sum_{i=1,\cdots,n-1} x_i}}\psi_{w_{n-1,\sum_{i=1,\cdots,n-2} -x_i}}\cdots\psi_{w_{3,-x_1-x_2}}\psi_{w_{2,-x_1}}(\xi(\sigma_n)) =$$

$$= \psi_{w_{1,\sum_{i=1,\cdots,n-1} x_i}}([x_n-(n-3)x_1-(n-4)x_2-\cdots-2x_{n-3}-x_{n-2},$$
$$,y,\ldots,y,-(n-2)y-x_n+(n-3)x_1+(n-4)x_2-\cdots+2x_{n-3}+x_{n-2}](1,n)) =$$

$$= [x_{n-1}+x_{n-2}+\cdots+x_1,y,\ldots,y,-x_{n-1}-x_{n-2}-\cdots-x_1]$$
$$[x_n-(n-3)x_1-(n-4)x_2-\cdots-2x_{n-3}-x_{n-2},y,\ldots,y,-(n-2)y-x_n+(n-3)x_1+(n-4)x_2-\cdots+2x_{n-3}+x_{n-2}](1,n)$$
$$[-x_{n-1}-x_{n-2}-\cdots-x_1,y,\ldots,y,x_{n-1}+x_{n-2}+\cdots+x_1] =$$

$$= [x_n-(n-4)x_1-(n-5)x_2-\cdots-x_{n-3}+x_{n-1},y,\ldots,y,-(n-2)y-x_n+(n-4)x_1+(n-5)x_2+\cdots+x_{n-3}-x_{n-1}](1,n)$$

$$[-x_{n-1} - x_{n-2} - \cdots - x_1, y, \ldots, y, x_{n-1} + x_{n-2} + \cdots + x_1] =$$

$$= [x_n - (n-5)x_1 - (n-6)x_2 - \cdots - x_{n-4} + 2x_{n-1} + x_{n-2},$$
$$, y, \ldots, y, -(n-2)y - x_n + (n-5)x_1 + (n-6)x_2 + \cdots + x_{n-4} - 2x_{n-1} - x_{n-2}](1, n).$$

So $p = x_n - (n-5)x_1 - (n-6)x_2 - \cdots - x_{n-4} + 2x_{n-1} + x_{n-2}$.

□

Remark 6.3. Proposition 6.2 states that instead of $n+1$ parameters in ξ, the x_i, for $i = 1, \ldots, n$, and y, we just have to deal with two parameters y and p. It does not matter the values of each individual x_i, it just matters the value of p. So from now on we will just deal with the morphisms $\xi_{y,p}$ with $y, p \in \mathbb{Z}$.

The graph automorphisms form a dihedral group generated by a rotation, ρ, of the graph by an angle of $\frac{2\pi}{n}$ and a symmetry.

The rotation ρ

In order to understand the action of ρ in $\mathbb{Z}^n$ we will use as generating set, for $\mathbb{Z}^n$, $\{w_i = w_{i,1}\}_{i=1,\ldots,n-1}$. Now we fix

$$s_i = (i, i+1) \text{ for } i = 1, \ldots, n-1$$

and

$$s_n = [1, 0, \ldots, 0, -1](1, n),$$

as generators of $W(\widetilde{A}_{n-1})$. The rotation ρ (see Figure 4), is defined by:

$$\rho(s_i) = s_{i+1} \text{ for } i \neq n;$$

$$\rho(s_n) = s_1.$$

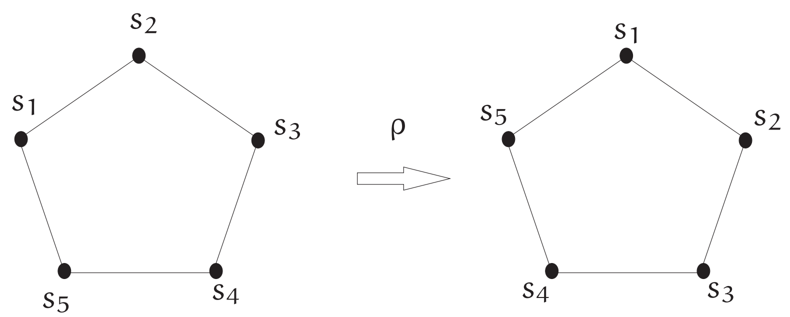

Figure 4: The rotation ρ acting on Coxeter graph $\tilde{A}_4$

Lemma 6.4. *Let $1 \leq k < n$. In the previous conditions we have*

$$w_1 = s_n s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1;$$

$$w_k = s_{k-1} \cdots s_1 w_1 s_1 \cdots s_{k-1}.$$

Proof. The case w_1 is just a direct computation using the fact that $s_1 \cdots s_{n-2}s_{n-1}s_{n-2}\cdots s_1 = (1,n)$. Suppose that

$w_{k-1} = s_{k-2}\cdots s_1 w_1 s_1 \cdots s_{k-2} = [\underbrace{0,\ldots,0}_{k-2},1,0,\ldots,0,-1]$.

Now we have that

$s_{k-1}w_{k-1}s_{k-1} = s_{k-1}s_{k-1}[\underbrace{0,\ldots,0}_{k-1},1,0,\ldots,0,-1] = w_k$.

□

Proposition 6.5. *Let $1 \le k < n$. In the previous conditions, we have:*

$$\rho(w_i) = w_{i+1}w_1^{-1} \text{ for } i = 1,\ldots,n-2;$$

and

$$\rho(w_{n-1}) = w_1^{-1}.$$

Proof. We will start by computing $\rho(w_i)$. By lemma 6.4 we have

$$w_1 = s_n s_1 \cdots s_{n-2}s_{n-1}s_{n-2}\cdots s_1,$$

so

$\rho(w_1) = s_1s_2\cdots s_{n-1}s_ns_{n-1}\cdots s_2 = (1,2)(2,3)\cdots(n-1,n)[1,0,\ldots,0,-1](1,n)$ $(n-1,n)\cdots(2,3) = (n,\ldots,1)[1,0,\ldots,0,-1](1,n)(2,\ldots,n) = (n,\ldots,1)[1,0,\ldots,0,-1](1,\ldots,n) = [-1,1,0,\ldots,0] = w_2w_1^{-1}$.

Suppose that $\rho(w_k) = w_{k+1}w_1^{-1}$ for $1 < k < n-1$. By lemma 6.4 we have

$\rho(w_{k+1}) = \rho(s_kw_ks_k) = s_{k+1}[-1,\underbrace{0,\ldots,0}_{k-1},1,0,\ldots,0]s_{k+1} =$

$= [-1,\underbrace{0,\ldots,0}_{k},1,0,\ldots,0] = w_{k+2}w_1^{-1}$.

Suppose that $k = n-1$. By lemma 6.4 we have

$\rho(w_{n-1}) = \rho(s_{n-2}w_{n-2}s_{n-2}) = s_{n-1}[-1,\underbrace{0,\ldots,0}_{n-3},1,0]s_{n-1} =$

$= [-1,0,\ldots,0,1] = w_1^{-1}$.

□

The symmetry γ

Now we analyze the other generating automorphism, the symmetry γ (see Figure 5), of $W(\widetilde{A}_{n-1})$. Let s_i, as before, denote the image of σ_i by the standard epimorphism. So

$$\gamma(s_1) = s_1;$$

and

$$\gamma(s_i) = s_{n-i+2} \text{ for } i \neq 1.$$

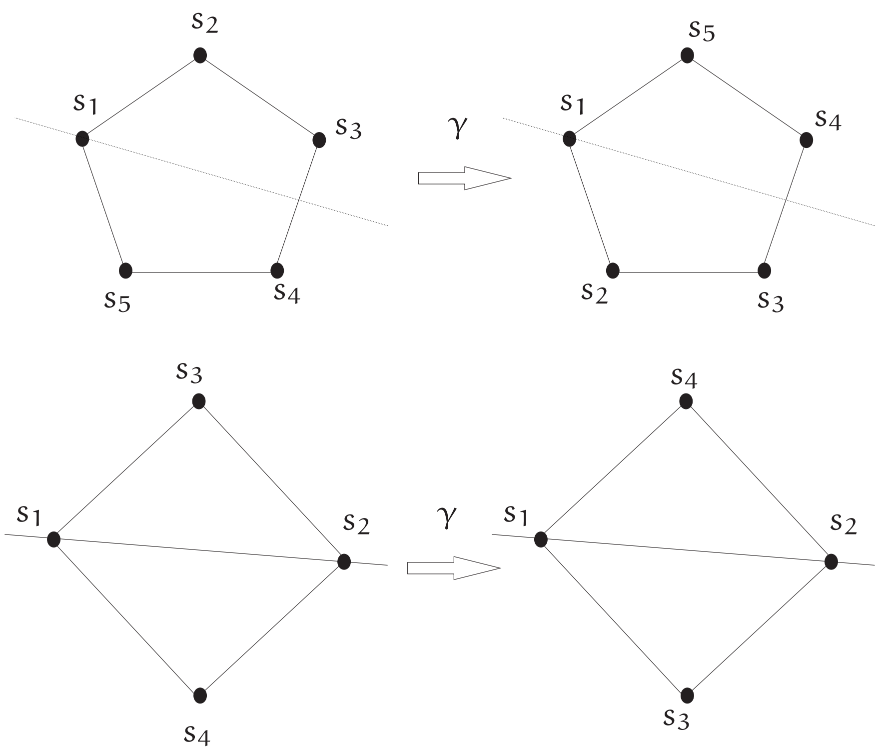

Figure 5: The symmetry γ acting on Coxeter graphs $\tilde{A}_4$ and $\tilde{A}_3$

Proposition 6.6. *In these conditions we have:*

$$\gamma(w_1) = w_2^{-1}w_3;$$

$$\gamma(w_2) = w_3w_1^{-1}$$

$$\gamma(w_3) = w_3;$$

$$\gamma(w_i) = w_3w_{n+3-i}^{-1}, \textit{ for } i = 4, \ldots, n-1;$$

Proof. We will start by computing $\gamma(w_i)$. By lemma 6.4 we have

$$w_1 = s_n s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1,$$

so for $i = 1$ we have

$$\gamma(w_1) = s_2 s_1 s_n s_{n-1} \cdots s_4 s_3 s_4 \cdots s_n s_1 =$$

$$= s_2 s_1 [1, 0, \ldots, 0, -1](1, n)(3, n)[1, 0, \ldots, 0, -1](1, n) s_1 =$$

$$= [0, 0, 1, 0 \ldots, 0, -1] s_2 s_1 (1, n)(3, n)[1, 0, \ldots, 0, -1](1, n) s_1 =$$

$$= [0, 0, 1, 0 \ldots, 0, -1] s_2 s_1 (1, n)[1, 0, -1, 0, \ldots, 0](3, n)(1, n) s_1 =$$

$$= [0, -1, 1, 0 \ldots, 0] s_2 s_1 (1, n)(3, n)(1, n) s_1 = w_2^{-1} w_3.$$

Let $i = 2$.

$$\gamma(w_2) = \gamma(s_1 w_1 s_1) = \gamma(s_1) w_2^{-1} w_3 \gamma(s_1) = w_3 w_1^{-1}.$$

Let $i = 3$ then

$$\gamma(w_3) = \gamma(s_2 w_2 s_2) = \gamma(s_2) w_3 w_1^{-1} \gamma(s_2) =$$
$$= s_n w_3 w_1^{-1} s_n = w_1 (1, n) w_3 (1, n) = w_3 (1, n)(1, n) = w_3.$$

Let $i = 4$ then

$$\gamma(w_4) = \gamma(s_3 w_3 s_3) = \gamma(s_3) w_3^{-1} \gamma(s_3) =$$
$$= s_{n-1} w_3^{-1} s_{n-1} = w_3^{-1} w_{n-1}.$$

Suppose that the formula is valid for $4 < i < n$. We have

$$\gamma(w_{i+1}) = \gamma(s_i w_i s_i) = \gamma(s_i) w_3 w_{n-i+3}^{-1} \gamma(s_i) =$$
$$= s_{n-i+2} w_3 w_{n-i+3}^{-1} s_{n-i+2} = w_3 w_{n-(i+1)+3}^{-1}.$$

□

7. Conditions to be an Epimorphism

In this section we will use the criterium of the previous section. We will obtain for each case l conditions on the parameters that tell us when ξ_l is an epimorphism. Let us fix $n > 2$.

7.1. Case l=1

We are working in $A(\widetilde{A}_{n-1})$ and we have to compute $\xi_{y,p}(\phi(\pi_1(p(\mathcal{G}), id))$. We will start by studding this set.

Lemma 7.1. *Let $y, p \in \mathbb{Z}$, $\sigma_i \in A(\widetilde{A}_{n-1})$, $1 \leq i \leq n-1$ we have:*

$$\xi_{y,p}(\sigma_k \cdots \sigma_1) = [\underbrace{-(n-1-k)y, \ldots, -(n-1-k)y}_{k}, 0, ky, \ldots, ky](1, \ldots, k+1)$$

and

$$\xi_{y,p}(\sigma_1 \cdots \sigma_k) = [-(nk-2k)y, \underbrace{(k-1)y, \ldots, (k-1)y}_{k}, ky, \ldots, ky](1, k+1, \ldots, 2)$$

for $k < n$.

Proof. By induction the case $k = 1$ being trivial let us look to the general case.

$\xi_{y,p}(\sigma_{k+1}\cdots\sigma_1) = \xi_{y,p}(\sigma_{k+1})\xi_{y,p}(\sigma_k\cdots\sigma_1) =$
$= [y,\ldots,y,-(n-2)y,0,y,\ldots,y](k+1,k+2)[\underbrace{-(n-1-k)y,\ldots,-(n-1-k)y}_{k},0,ky,\ldots,ky](1,\ldots,k+1) =$
$= [\underbrace{-(n-1-(k+1))y,\ldots,-(n-1-(k+1))y}_{k+1},0,(k+1)y,\ldots,(k+1)y](1,\ldots,k+2)$

and

$\xi_{y,p}(\sigma_1\cdots\sigma_{k+1}) = \xi_{y,p}(\sigma_1\cdots\sigma_k)\xi_{y,p}(\sigma_{k+1}) =$
$= [-(nk-2k)y,\underbrace{(k-1)y,\ldots,(k-1)y}_{k},ky,\ldots,ky](1,k+1,\ldots,2)[y,\ldots,y,-(n-2)y,0,y,\ldots,y](k+1,k+2) =$
$= [-(n(k+1)-2(k+1))y,\underbrace{ky,\ldots,ky}_{k+1},(k+1)y,\ldots,(k+1)y](1,k+2,\ldots,2).$

□

Lemma 7.2. *Let $y,p \in \mathbb{Z}$, $\sigma_i \in A(\widetilde{A}_{n-1})$, $1 \le i \le n-1$ we have:*

$$\xi_{y,p}((\sigma_k\cdots\sigma_1)^{-1}) = [0,\underbrace{(n-1-k)y,\ldots,(n-1-k)y}_{k},-ky,\ldots,-ky](1,k+1,\ldots,2)$$

and

$$\xi_{y,p}((\sigma_1\cdots\sigma_k)^{-1}) = [\underbrace{-(k-1)y,\ldots,-(k-1)y}_{k},(nk-2k)y,-ky,\ldots,-ky](1,\ldots,k+1)$$

for $k < n$.

Proof. Notice that $\xi_{y,p}((\sigma_k\cdots\sigma_1)^{-1})\xi_{y,p}(\sigma_k\cdots\sigma_1) = \xi_{y,p}(\sigma_k\cdots\sigma_1)\xi_{y,p}((\sigma_k\cdots\sigma_1)^{-1}) = [0,\ldots,0]id$ and $\xi_{y,p}((\sigma_1\cdots\sigma_k)^{-1})\xi_{y,p}(\sigma_1\cdots\sigma_k) = \xi_{y,p}(\sigma_1\cdots\sigma_k)\xi_{y,p}((\sigma_1\cdots\sigma_k)^{-1}) = [0,\ldots,0]id.$

□

Lemma 7.3. *Let g be a generator of $P(A(A_{n-1}))$. Then $\xi_{y,p}(g) = [Q_1(y),\ldots,Q_n(y)]id$ for some homogeneous polynomials Q_i.*

Proof. This result is trivial by the preceding lemmas. We will write a proof of it to be able to obtain more information about the polynomials Q_i. It is well known (see [B]) that the elements

$$a_{ij} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$$

generate the pure braid group $P(A(A_{n-1}))$. So let us compute $\xi_{y,p}(a_{ij})$ for all $i < j$:

$\xi_{y,p}(a_{ij}) = \xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1}\xi_{y,p}(\sigma_i^2)(\xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1})^{-1}$

So we start by computing

$\xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1} = [\underbrace{-(n-2-j)y,\ldots,-(n-2-j)y}_{j-1},0,(j-1)y,\ldots,(j-1)y](1,\ldots,j)[0,\underbrace{(n-1-i)y,\ldots,(n-1-i)y}_{i},-iy,\ldots,-iy](1,i+1,\ldots,2) =$

$= [\underbrace{(j-i+1)y,\dots,(j-i+1)y}_{i}, \underbrace{-(n-2-j+i)y,\dots,-(n-2-j+i)y}_{j-i-1}, 0, (j-1-i)y,\dots,(j-1-i)y](i+1,\dots,j)$.

now we have

$(\xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1})^{-1} = [-(j-i+1)y,\dots,-(j-i+1)y, 0, \underbrace{(n-2-j+i)y,\dots,(n-2-j+i)y}_{j-i-1}, -(j-1-i)y,\dots,-(j-1-i)y](i+1,j,\dots,i+2)$.

For the missing factor it is easy to compute and we have:

$\xi_{y,p}(\sigma_i^2) = [y,\dots,y,-(n-2)y,0,y,\dots,y](i+1,i+2)[y,\dots,y,-(n-2)y,0,y,\dots,y](i,i+1) =$
$= [2y,\dots,2y,-(n-2)y,-(n-2)y,2y,\dots,2y]id$.

So

$\xi_{y,p}(a_{ij}) = \xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1}\xi_{y,p}(\sigma_i^2)(\xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1})^{-1} =$
$= [\underbrace{(j-i+3)y,\dots,(j-i+3)y}_{i-1}, (j-i+3-n)y, \underbrace{-(n-4-j+i)y,\dots,-(n-4-j+i)y}_{j-i-1}, -(n-2)y, \underbrace{(j-i+1)y,\dots,(j-i+1)y}_{n-j+1}](i+1,\dots,j)(\xi_{y,p}(\sigma_{j-1}\cdots\sigma_1)(\xi_{y,p}(\sigma_i\cdots\sigma_1))^{-1})^{-1} =$

$= [\overbrace{\underbrace{2y,\dots,2y}_{i-1}, -(n-2)y, 2y,\dots,2y}^{j-1}, -(n-2)y, 2y,\dots,2y]id$

∎

Remark 7.4. We notice, from the proof of lemma 7.3 that not only $\xi_{y,p}(g) = [Q_1(y),\dots,Q_n(y)]id$ for some homogeneous polynomials Q_i, but in the case were n is even, we have

$$\xi_{y,p}(g) = [Q'_1(2y),\dots,Q'_n(2y)]id.$$

Lemma 7.5. *Let $g \in \phi(\pi_1(p(\mathcal{G}), id))$ be a generator such that*

$$g = s\sigma_n t^{-1}.$$

For some simple elements $s, t \in A_{n-1}$. Then

$$\xi_{y,p}(g) = [Q_1(y,p),\dots,Q_n(y,p)]id.$$

Were Q_i are homogeneous polynomials with integer coefficients.

Proof. By Lemma 5.2 we know that it suffices to check this type of generators. We will identify s and t with its images in S_n. Now suppose that $l(s) \geq l(t)$, we will proceed by induction on $l(t)$.

If $l(t) = 0$ then $s = \sigma_1\cdots\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ which is the simple element with associated permutation $(1,n)$. We have

$\xi_{y,p}(g) = \xi_{y,p}(s)\xi_{y,p}(\sigma_n) = \xi_{y,p}(\sigma_1\cdots\sigma_{n-1})\xi_{y,p}(\sigma_{n-2}\cdots\sigma_1)\xi_{y,p}(\sigma_n) =$
$= [-(n(n-1)-2(n-1))y,(n-2)y,\dots,(n-2)y](1,n,\dots,2)[\underbrace{-y,\dots,-y}_{n-2},0,(n-2)y](1,\dots,n-1)\xi_{y,p}(\sigma_n) =$
$= [-(n(n-1)-(3n-4))y,(n-3)y,\dots,(n-3)y,(n-2)y](1,n)[p,y,\dots,y,-(n-2)y-p](1,n) =$
$= [-(n(n-1)-2(n-1))y-p,(n-2)y,\dots,(n-2)y,(n-2)y+p]id.$

If the result is true for $l(t) = r$, suppose that σ_k is a initial letter of t, then we write:

$$g = s\sigma_n t^{-1} = s\sigma_n t_1^{-1}\sigma_k^{-1}.$$

Suppose that σ_k is also an initial letter of s, then

$$g = \sigma_k s_1 \sigma_n t_1^{-1} \sigma_k^{-1}.$$

Now we know that $\xi_{y,p}(s_1\sigma_n t_1^{-1}) = [Q_1(y,p), \ldots, Q_n(y,p)]id$ by hypothesis and so $\xi_{y,p}(g) = [\underbrace{y, \ldots, y, -(n-2)y}_{k}, 0, y, \ldots, y](k, k+1)[Q_1(y,p), \ldots, Q_n(y,p)]id[\underbrace{-y, \ldots, -y, 0}_{k}, (n-2)y, -y, \ldots, -y](k, k+1) =$
$= [y + Q_1(y,p), \ldots, y + Q_{k-1}(y,p), -(n-2)y + Q_{k+1}(y,p), Q_k(y,p), y + Q_{k+2}(y,p), \ldots, y + Q_n(y,p)](k, k+1)[-y, \ldots, -y, 0, (n-2)y, -y, \ldots, -y](k, k+1) =$
$= [Q_1(y,p), \ldots, Q_{k-1}(y,p), Q_{k+1}(y,p), Q_k(y,p), Q_{k+2}(y,p), \ldots, Q_n(y,p)]id.$

Suppose now that σ_k is not an initial letter of s, then

$$g = \sigma_k^{-1} \sigma_k s_1 \sigma_n t_1^{-1} \sigma_k^{-1}.$$

So we have $\sigma_k s_1$ is a simple element hence $\sigma_k s_1 \sigma_n t_1^{-1}$ is a generator and $\xi_{y,p}(\sigma_k s_1 \sigma_n t_1^{-1}) = [Q_1(y,p), \ldots, Q_n(y,p)]id$ and $\xi_{y,p}(g) = [\underbrace{-y, \ldots, -y}_{k-1}, 0, (n-2)y, -y, \ldots, -y](k, k+1)[Q_1(y,p), \ldots, Q_n(y,p)]id[\underbrace{-y, \ldots, -y}_{k-1}, 0, (n-2)y, -y, \ldots, -y](k, k+1) =$
$= [-y + Q_1(y,p), \ldots, -y + Q_{k-1}(y,p), Q_{k+1}(y,p), (n-2)y + Q_k(y,p), -y + Q_{k+2}(y,p), \ldots, -y + Q_n(y,p)](k, k+1)[\underbrace{-y, \ldots, -y}_{k-1}, 0, (n-2)y, -y, \ldots, -y](k, k+1) =$
$= [-2y + Q_1(y,p), \ldots, -2y + Q_{k-1}(y,p), (n-2)y + Q_{k+1}(y,p), (n-2)y + Q_k(y,p), -2y + Q_{k+2}(y,p), \ldots, -2y + Q_n(y,p)]id.$

□

Remark 7.6. We notice, from the proof of lemma 7.5 that not only $\xi_{y,p}(g) = [Q_1(y,p), \ldots, Q_n(y,p)]id$ for some homogeneous polynomials Q_i, but in the case were n is even, we have

$$\xi_{y,p}(g) = [Q'_1(2y,p), \ldots, Q'_n(2y,p)]id.$$

Proposition 7.7. *Let $g \in \phi(\pi_1(p(\mathcal{G}), id))$ then*

$$\xi_{y,p}(g) = [Q_1(y,p), \ldots, Q_n(y,p)]id.$$

Were Q_i are homogeneous polynomials with integer coefficients.

Proof. It is obvious that we can suppose that g is a generator of $\phi(\pi_1(p(\mathcal{G}), id)$. As we saw in section 3 there are two types of generator, those appearing as generators of $P(A(A_{n-1}))$ and the ones in which we have the occurrence of σ_n. The cases are both treated in lemmas 7.3 and 7.5 respectively. □

Now the first result concerning the parameters is:

Proposition 7.8. *If $gcd(y,p) \neq 1$ then $\xi_{y,p}$ is not an epimorphism.*

Proof. Suppose that $gcd(y,p) = q \neq 1$. By proposition 7.7 we can conclude that $Im(\xi_{y,p}) \subset (q\mathbb{Z})^n$. □

For the rest of this section we will suppose that $gcd(y, p) = 1$.

In order to use the criterium described above, it is sufficient to verify it for a submatrix of M. In order to do so we will choose some particular elements in the image of $\xi_{y,p}$ and show after that this submatrix, on the first $n-1$ columns, verify the hypothesis of the criterium.

Let us define:

$$g_k = \sigma_{k-1} \cdots \sigma_1 \sigma_{k+1} \cdots \sigma_n (\sigma_k \cdots \sigma_1 \sigma_{k+1} \cdots \sigma_{n-1})^{-1} \text{ for } k = 1, \dots, n-1;$$
$$g_n = \sigma_n^2;$$
$$g_{n+i} = \sigma_i^2.$$

Lemma 7.9. *In the previous conditions we have:*

$$\xi_{y,p}(g_k) = [\underbrace{0, \dots, 0}_{k-1}, p + (n^2 - nk - n)y, -p - (n^2 - nk - n)y, 0, \dots, 0]id.$$

Proof. This proof is a simple and direct computation using lemmas 7.1 and 7.2. Let us decompose

$$g_k = \sigma_{k-1} \cdots \sigma_1 (\sigma_1 \cdots \sigma_k)^{-1} (\sigma_1 \cdots \sigma_{n-1}) \sigma_n (\sigma_1 \cdots \sigma_{n-1})^{-1} (\sigma_1 \cdots \sigma_k)(\sigma_k \cdots \sigma_1)^{-1}.$$

Assume for now that k is even. We will start by computing

$\sigma_{k-1} \cdots \sigma_1 (\sigma_1 \cdots \sigma_k)^{-1} (\sigma_1 \cdots \sigma_{n-1}) \sigma_n =$

$= [\underbrace{-(n-1)y, \dots, -(n-1)y}_{k-1}, -(k-1)y, (nk-k-1)y, -y, \dots, -y](1, 3, \dots, k+1)(2, 4, \dots, k)(\sigma_1 \cdots \sigma_{n-1})\sigma_n =$

$= [\underbrace{-y, \dots, -y}_{k-1}, (n-k-1)y, (-n^2+3n-k+nk-3)y, \underbrace{(n-3)y, \dots, (n-3)y}_{n-k-1}](1, \dots, k)(n, \dots, k+1)\sigma_n =$

$= [\underbrace{0, \dots, 0}_{k-1}, (n-k-1)y+p, -p+(-n^2+2n-k+nk-1)y, \underbrace{(n-2)y, \dots, (n-2)y}_{n-k-1}](1, \dots, k, n, \dots_k+1).$

Assume for now that k is odd the only change is in step 1, we must replace the permutation $(1, 3, \dots, k+1)(2, 4, \dots, k)$ by $(1, 3, \dots, k, 2, 4, \dots, k+1)$ but after we obtain exactly the same result.

Now we compute

$(\sigma_1 \cdots \sigma_{n-1})^{-1} (\sigma_1 \cdots \sigma_k)(\sigma_k \cdots \sigma_1)^{-1} =$

$= [\underbrace{-(n-k-1)y, \dots, -(n-k-1)y}_{k}, \underbrace{-(n-k-2)y, \dots, -(n-k-2)y}_{n-k-1}, (n^2-3n+2-nk-2k)y](k+1, \dots, n)(\sigma_k \cdots \sigma_1)^{-1} =$

$= [-(n-k-1)y, \underbrace{0, \dots, 0}_{k-1}, -(n-2)y, \dots, -(n-2)y, (n^2-2n-nk+2k+1)y](k+1, \dots, n, k, \dots, 1).$

So we have our result by multiplying these to elements in order to obtain $\xi_{y,p}(g_k)$. ∎

A direct consequence of Lemma 7.9 is:

Lemma 7.10. *For $i = 1, \dots, n-1$ we have:*

$$\xi_{y,p}(g_n) = [-(n-2)y, 2y, \dots, 2y, -(n-2)y]id.$$

$$\xi_{y,p}(g_{n+i}) = [\underbrace{2y, \dots, 2y}_{i-1}, -(n-2)y, -(n-2)y, 2y, \dots, 2y]id.$$

We must now compute some determinants in order to prove later that their gcd is 1. Using as a list of generators $g_1, \dots, g_{2n-1}$ defined above we have:

Lemma 7.11. *Let $k = 1, \dots, n-1$, then*

$$gcd(y, p - (n^2 - nk - n)y) = 1.$$

Proof. Suppose that we have $p - (n^2 - nk - n)y = qt_1$ and $y = qt_2$ with $q > 1$. Then $p = q(t_2 + (n^2 - nk - n)t_1)$ meaning that $gcd(p, y) \geq q$ which is false.

□

Lemma 7.12.

$$det_{1,\dots,n-1} = \prod_{k=1,\dots,n-1} (p - (n^2 - kn - n)y);$$

$$det_{n+2,2,\dots,n-1} = 2y \prod_{k=2,\dots,n-1} (p - (n^2 - kn - n)y);$$

$$det_{n+1,2,\dots,n-1} = (n-2)y \prod_{k=2,\dots,n-1} (p - (n^2 - kn - n)y).$$

Proposition 7.13. *Let $y = 0$ and $p = \pm 1$. Then $\xi_{y,p}$ is an epimorphism.*

Proof. By lemma 7.12 we have that $det_{1,\dots,n-1} = \pm 1$. By proposition 4.2 we are done.

□

Lemma 7.14. *Let n be odd and $y \neq 0$. There exist an integer $C \neq 0$ such that*

$$det_{n+1,\dots,2n-1} = Cy^{n-1}.$$

Proof. We will show that the matrix formed by the $\xi_{y,p}(g_{n+i})$ (see Lemmas 7.10), with $i = 1, \dots, n-1$ as $rank(n-1)$.

$$R = \begin{bmatrix} -(n-2)y & -(n-2)y & 2y & \cdots & \cdots & \cdots & 2y \\ 2y & -(n-2)y & -(n-2)y & 2y & \cdots & \cdots & 2y \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ 2y & \cdots & \cdots & \cdots & -(n-2)y & -(n-2)y & 2y \\ 2y & \cdots & \cdots & \cdots & 2y & -(n-2)y & -(n-2)y \end{bmatrix}$$

Now the first row cannot be a linear (with integer coefficients) combination of the remaining $n-2$ rows, because that would give us, using the first column the equation

$$-(n-2)y = 2ky, \text{ for some } k \in \mathbb{Z}$$

which implies that $n-2$ is even. So now we repeat the process to show that the second row cannot be a linear (with integer coefficients) combination of the remaining $n-3$ rows. In the end we conclude that we obtain $n-1$ independent rows, so $rank(R) = n-1$ and we are done.

Now we know that $det(R) = Cy^{n-1}$ with $C \neq 0$

□

Proposition 7.15. *If n is odd then $\xi_{y,p}$ is an epimorphism.*

Proof. Using lemmas 7.11, 7.12 and 7.12 we may compute gcd of $det_{1,\ldots,n-1}$, $det_{n+1,\ldots,2n-1}$, $det_{n+1,2,\ldots,n-1}$ and $det_{n+2,2,\ldots,n-1}$.

□

Proposition 7.16. *If n is even and p is odd, then $\xi_{y,p}$ is an epimorphism.*

Proof. It remains to see which is $gcd(2, p-(n^2-nk-n)y$ for all $k=1,\ldots,n-1$. But 2 divide $(n^2-nk-n) = n(n-k-1)$ and not p so it cannot divide $p-(n^2-nk-n)y$. We can conclude that $gcd(2, p-(n^2-nk-n)y = 1$.

□

Proposition 7.17. *If n is even and p is even, then $\xi_{y,p}$ is not an epimorphism.*

Proof. By remarks 7.4 and 7.6 we have in this case that all entries of the matrix are multiples of 2 and so $\xi_{y,p}$ is not an epimorphism.

□

Recall the definition of $\xi_{y,p}$ from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$, $n>2$:

$$\xi_{y,p}(\sigma_i) = [\underbrace{y,\ldots,y}_{i-1}, -(n-2)y, 0, y, \ldots, y](i, i+1), \quad 1 \le i < n$$

and

$$\xi_{y,p}(\sigma_n) = [p, y, \ldots, y, -(n-2)y - p](1, n).$$

We can now state the main result proved in this section:

Proposition 7.18. *Let $y, p \in \mathbb{Z}$ and $gcd(y,p) = 1$.*

1. *If $n \ge 3$ is odd then $\xi_{(y,p)}$ is an epimorphism.*
2. *If $n \ge 3$ is even and p is odd then $\xi_{(y,p)}$ is an epimorphism.*

7.2. Case l=2

Proposition 7.19. *The morphism ξ it is not an epimorphism for all $x_i \in \mathbb{Z}$.*

Proof. The image by ξ of all elements in $\phi(\pi_1(p(\mathcal{G}), id)$ is $[0,0,0,0,0,0]id$.

□

7.3. Case l=3

When computing $\phi(\pi_1(p(\mathcal{G}), id))$ we obtain only 3 distinct elements different from zero:

$$[-x_1-x_2, -x_2-x_3, x_1+x_2, x_2+x_3]id,$$

$$[-x_2, -x_1-x_2-x_3, x_1+x_2+x_3, x_2]id,$$

$$[x_3, -x_3, x_1, -x_1]id.$$

Proposition 7.20. *The morphism ξ is an epimorphism if and only if*

$$det_{1,2,3} = (x_3-x_1)(x_3+x_1)(x_3+x_1+2x_2) = \pm 1.$$

So we have 8 epimorphisms, $\xi_{(\pm 1,0,0)}$, $\xi_{(\pm 1,\mp 1,0)}$, $\xi_{(0,0,\pm 1)}$ and $\xi_{(0,\pm 1,\mp 1)}$.

7.4. Case l=4

When computing $\phi(\pi_1(p(\mathcal{G}), id))$ we obtain only 3 distinct elements different from zero:

$$[-x_3 - x_1, -x_2 - x_3, x_2 + x_3, x_3 + x_1]id,$$
$$[-x_3 - x_2 - x_1, -x_3, x_3 + x_2 + x_1, x_3]id,$$
$$[-x_1, x_1, x_2, -x_2]id.$$

Proposition 7.21. *The morphism ξ is an epimorphism if and only if*

$$det_{1,2,3} = (x_2 - x_1)(x_2 + x_1)(x_2 + x_1 + 2x_3) = \pm 1.$$

So we have 8 epimorphisms, $\xi_{(0,\pm1,0)}$, $\xi_{(0,\pm1,\mp1)}$, $\xi_{(\pm1,0,0)}$ and $\xi_{(\pm1,0,\mp1)}$.

7.5. Case l=5

This case derives from the general case so the images of σ_1, σ_2 and σ_3 are as in the general case and the image of σ_n is equal to the image of σ_2. This means also that we have one less parameter.

When computing $\phi(\pi_1(p(\mathcal{G}), id))$ we obtain only 15 distinct elements different from zero, that are displayed in the following matrix:

$$M = \begin{bmatrix} -x_1 - x_2 - x_3 - x_4 & -3x_4 - x_2 & x_2 - x_4 & 5x_4 + x_3 + x_2 + x_1 \\ 2x_4 & 2x_4 & -2x_4 & -2x_4 \\ -x_1 - x_2 & -4x_4 - x_2 - x_3 & 4x_4 + x_2 + x_1 & x_2 + x_3 \\ 2x_4 & -2x_4 & -2x_4 & 2x_4 \\ 2x_4 & -2x_4 & 2x_4 & -2x_4 \\ x_4 - x_1 & 3x_4 + x_1 & -3x_4 - x_3 & x_3 - x_4 \\ -2x_4 & -2x_4 & 2x_4 & 2x_4 \\ -4x_4 - x_1 - x_2 & -x_3 - x_2 & x_2 + x_1 & 4x_4 + x_3 + x_2 \\ -5x_4 - x_1 - x_2 - x_3 & x_4 - x_2 & 3x_4 + x_2 & x_2 + x_3 + x_4 + x_1 \\ -2x_4 & 2x_4 & -2x_4 & 2x_4 \\ -2x_4 & 2x_4 & 2x_4 & -2x_4 \\ -x_4 - x_1 & x_1 + x_4 & -x_4 - x_3 & x_4 + x_3 \\ -3x_4 - x_1 & x_1 - x_4 & x_4 - x_3 & 3x_4 + x_3 \\ -3x_4 - x_1 - x_2 - x_3 & -x_4 - x_2 & x_4 + x_2 & 3x_4 + x_3 + x_2 + x_1 \\ -x_1 - 2x_4 - x_2 & -2x_4 - x_2 - x_3 & x_2 + x_1 + 2x_4 & 2x_4 + x_3 + x_2 \end{bmatrix}$$

corresponding the line i with the image of the generator g_i.

Proposition 7.22. *If $gcd(x_1, \ldots, x_4) \neq 1$ then ξ is not an epimorphism.*

Proof. It is sufficient to notice that in this case all det_{i_1,i_2,i_3} are multiples of $gcd(x_1, \ldots, x_4)$ for all $i_j \in \{1, \ldots, 15\}$, $j \in \{1, 2, 3\}$ and $i_{j_1} \neq i_{j_2}$ if $j_1 \neq j_2$. $\square$

Suppose from now on that $gcd(x_1, \ldots, x_4) = 1$.

We compute all different det_{i_1,i_2,i_3}.

$det_{1,2,3} = -2x_4(2x_4x_3 + 16x_4^2 - 2x_4x_1 - x_1^2 + x_3^2)$,
$det_{1,2,4} = -8x_4^2(x_3 + x_1 + 2x_4)$,
$det_{1,2,5} = -32x_4^3$,
$det_{1,2,6} = 2x_4(x_3 + x_1 + 2x_4)(x_1 + 4x_4 - x_3)$,

$det_{1,2,8} = 2x_4(16x_4^2 - 2x_4x_3 + 2x_4x_1 + x_1^2 - x_3^2),$
$det_{1,2,9} = 16x_4^2(x_3 + x_1 + 2x_4),$
$det_{1,2,10} = 32x_4^3,$
$det_{1,2,11} = 8x_4^2(x_3 + x_1 + 2x_4),$
$det_{1,2,15} = -2x_4(x_3 - x_1)(x_3 + x_1 + 2x_4),$
$det_{1,3,4} = 2x_4(6x_4 + x_3 + x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,5} = 2x_4(x_1 + 4x_4 - x_3)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,6} = 16x_4^3 - 8x_4^2x_2 + x_1x_3^2 + 2x_2x_3^2 + x_3^3 - x_1^3 - 32x_4^2x_1 + 8x_4^2x_3 - 2x_2x_1^2 - x_1^2x_3 + 8x_4x_3^2 - 12x_4x_1^2 - 4x_4x_1x_3 + 4x_2x_4x_3 - 12x_2x_4x_1,$
$det_{1,3,8} = -4x_4(x_1 + 4x_4 - x_3)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,9} = -4x_4(6x_4 + x_3 + x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,10} = -2x_4(x_1 + 4x_4 - x_3)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,11} = -2x_4(6x_4 + x_3 + x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,12} = -(4x_4^2 - x_3^2 + x_1^2 - 2x_4x_3 + 6x_4x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,3,13} = -48x_4^3 - 8x_4^2x_2 + x_1x_3^2 + 2x_2x_3^2 + x_3^3 - x_1^3 - 24x_4^2x_1 - 2x_2x_1^2 - x_1^2x_3 + 4x_4x_3^2 - 8x_4x_1^2 - 4x_4x_1x_3 + 4x_2x_4x_3 - 12x_2x_4x_1,$
$det_{1,4,5} = -8x_4^2(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,4,6} = 2x_4(x_3 + x_1 + 2x_4)(x_1 + 8x_4 + x_3 + 2x_2),$
$det_{1,4,8} = 2x_4(2x_4 - x_3 - x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,4,10} = 8x_4^2(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,4,12} = 2x_4(x_3 + x_1 + 2x_4)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,4,13} = 2x_4(x_3 + x_1 + 2x_4)(2x_2 + x_3 + x_1),$
$det_{1,4,15} = -2x_4(x_3 + x_1 + 2x_4)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,5,6} = 2x_4(16x_4^2 - 4x_4x_1 + 4x_4x_3 + 2x_2x_3 - x_1^2 + x_3^2 - 2x_2x_1),$
$det_{1,5,9} = -16x_4^2(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,5,12} = 2x_4(x_3 - x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,5,13} = -2x_4(16x_4^2 + 4x_4x_1 - 4x_4x_3 - 2x_2x_3 + x_1^2 - x_3^2 + 2x_2x_1),$
$det_{1,6,8} = 48x_4^3 + 8x_4^2x_2 - x_1x_3^2 - 2x_2x_3^2 - x_3^3 + x_1^3 + 16x_4^2x_1 + 8x_4^2x_3 + 2x_2x_1^2 + x_1^2x_3 - 4x_4x_3^2 + 8x_4x_1^2 + 4x_4x_1x_3 + 4x_2x_4x_3 + 4x_2x_4x_1,$
$det_{1,6,9} = 4x_4(x_3 + x_1 + 2x_4)(x_1 + 8x_4 + x_3 + 2x_2),$
$det_{1,6,13} = 4x_4(x_3 + x_1 + 2x_4)(x_1 + 4x_4 - x_3),$
$det_{1,6,15} = (x_3 + x_1 + 2x_4)(x_1^2 + 8x_4x_1 + 2x_2x_1 - x_3^2 + 8x_4^2 - 4x_4x_3 + 4x_4x_2 - 2x_2x_3),$
$det_{1,7,8} = -2x_4(16x_4^2 - 2x_4x_3 + 2x_4x_1 + x_1^2 - x_3^2),$
$det_{1,7,9} = -16x_4^2(x_3 + x_1 + 2x_4),$
$det_{1,7,12} = -2x_4(x_3 + x_1 + 2x_4)(x_1 + 4x_4 - x_3),$
$det_{1,7,15} = 2x_4(x_3 - x_1)(x_3 + x_1 + 2x_4),$
$det_{1,8,9} = 4x_4(2x_4 - x_3 - x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,8,12} = -(4x_4^2 - x_3^2 + x_1^2 + 2x_4x_3 + 2x_4x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,8,13} = 16x_4^3 - 8x_4^2x_2 + x_1x_3^2 + 2x_2x_3^2 + x_3^3 - x_1^3 - 8x_4^2x_1 - 16x_4^2x_3 - 2x_2x_1^2 - x_1^2x_3 - 4x_4x_1^2 - 4x_4x_1x_3 - 4x_2x_4x_3 - 4x_2x_4x_1,$
$det_{1,9,12} = -4x_4(x_3 + x_1 + 2x_4)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,9,13} = -4x_4(x_3 + x_1 + 2x_4)(2x_2 + x_3 + x_1),$
$det_{1,9,15} = 4x_4(x_3 + x_1 + 2x_4)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,10,12} = -2x_4(x_3 - x_1)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,10,13} = 2x_4(16x_4^2 + 4x_4x_1 - 4x_4x_3 - 2x_2x_3 + x_1^2 - x_3^2 + 2x_2x_1),$
$det_{1,11,13} = -2x_4(x_3 + x_1 + 2x_4)(2x_2 + x_3 + x_1),$
$det_{1,12,15} = (x_3 + x_1 + 2x_4)(-x_3 + x_1 + 2x_4)(4x_4 + x_3 + x_1 + 2x_2),$
$det_{1,13,15} = (x_3 + x_1 + 2x_4)(x_1^2 + 4x_4x_1 + 2x_2x_1 + 4x_4x_2 - x_3^2 + 8x_4^2 - 2x_2x_3),$

$det_{2,3,5} = 8x_4^2(x_3 - x_1)$,
$det_{2,3,6} = -2x_4(x_3 - x_1)(6x_4 + x_3 + x_1)$,
$det_{2,3,8} = -16x_4^2(x_3 - x_1)$,
$det_{2,3,10} = -8x_4^2(x_3 - x_1)$,
$det_{3,4,6} = -2x_4(8x_4^2 - 6x_4x_3 - 6x_4x_1 - 4x_4x_2 - 2x_1x_3 - x_3^2 - 2x_2x_3 - x_1^2 - 2x_2x_1)$,
$det_{3,4,8} = 16x_4^2(4x_4 + x_3 + x_1 + 2x_2)$,
$det_{3,4,13} = 2x_4(24x_4^2 + 6x_4x_1 + 6x_4x_3 + 4x_4x_2 + x_1^2 + 2x_1x_3 + x_3^2 + 2x_2x_3 + 2x_2x_1)$,
$det_{5,6,8} = 2x_4(x_3 - x_1)(x_1 + 8x_4 + x_3 + 2x_2)$,
$det_{5,6,13} = 16x_4^2(x_3 - x_1)$,
$det_{6,7,8} = -2x_4(x_3 - x_1)(2x_4 - x_3 - x_1)$,
$det_{6,7,9} = -2x_4(x_3 + 4x_4 - x_1)(x_3 + x_1 + 2x_4)$,
$det_{7,8,9} = 2x_4(2x_4x_3 + 16x_4^2 - 2x_4x_1 - x_1^2 + x_3^2)$,
$det_{8,9,10} = -2x_4(x_3 + 4x_4 - x_1)(4x_4 + x_3 + x_1 + 2x_2)$,
$det_{8,9,11} = -2x_4(2x_4 - x_3 - x_1)(4x_4 + x_3 + x_1 + 2x_2)$,
$det_{8,9,12} = -(4x_4^2 + x_3^2 - x_1^2 + 2x_4x_3 + 2x_4x_1)(4x_4 + x_3 + x_1 + 2x_2)$,
$det_{8,9,13} = 16x_4^3 - 8x_4^2x_2 - x_1x_3^2 - 2x_2x_3^2 - x_3^3 + x_1^3 - 16x_4^2x_1 - 8x_4^2x_3 + 2x_2x_1^2 + x_1^2x_3 - 4x_4x_3^2 - 4x_4x_1x_3 - 4x_2x_4x_3 - 4x_2x_4x_1$,
$det_{8,9,15} = 2x_4(x_3 + 4x_4 - x_1)(4x_4 + x_3 + x_1 + 2x_2)$,
$det_{9,10,13} = -2x_4(16x_4^2 - 4x_4x_1 + 4x_4x_3 + 2x_2x_3 - x_1^2 + x_3^2 - 2x_2x_1)$,
$det_{13,14,15} = (x_3 - x_1)(x_3 + x_1 + 2x_4)(x_1 + 2x_4 + x_3 + 2x_2)$.

Proposition 7.23. *If $gcd(x_1, x_3, x_4) = k > 1$ then ξ is not an epimorphism.*

Proof. It is sufficient to notice that in this case all det_{i_1,i_2,i_3} are multiples of k.

□

Proposition 7.24. *If $x_1 + x_3$ is even then ξ is not an epimorphism.*

Proof. It is sufficient to notice that in this case all det_{i_1,i_2,i_3} are even.

□

Let us denote by $p = x_1 + x_3 + 2x_4$ and $q = x_1 - x_3$.

Now we rewrite the above equations:

$det_{1,2,3} = 2x_4(-16x_4^2 + qp)$;
$det_{1,2,4} = -8x_4^2p$;
$det_{1,2,5} = -32x_4^3$;
$det_{1,2,6} = 2x_4p(4x_4 + q)$;
$det_{1,2,8} = 2x_4(16x_4^2 + qp)$;
$det_{1,2,9} = 16x_4^2p$;
$det_{1,2,10} = 32x_4^3$;
$det_{1,2,11} = 8x_4^2p$;
$det_{1,2,15} = 2x_4qp$;
$det_{1,3,4} = 2x_4(4x_4 + p)(2x_4 + p + 2x_2)$;
$det_{1,3,5} = 2x_4(4x_4 + q)(2x_4 + p + 2x_2)$;
$det_{1,3,6} = -4x_4x_2q - 4x_4x_2p - 6x_4qp - 4x_4^2q - 4x_4^2p - 2x_4p^2 - 2x_2qp - qp^2 + 32x_4^3$;
$det_{1,3,8} = -4x_4(4x_4 + q)(2x_4 + p + 2x_2)$;
$det_{1,3,9} = -4x_4(4x_4 + p)(2x_4 + p + 2x_2)$;
$det_{1,3,10} = -2x_4(4x_4 + q)(2x_4 + p + 2x_2)$;
$det_{1,3,11} = -2x_4(4x_4 + p)(2x_4 + p + 2x_2)$;
$det_{1,3,12} = -(2qx_4 + 2x_4p + qp)(2x_4 + p + 2x_2)$;

$det_{1,3,13} = -4x_4x_2q - 4x_4x_2p - 2x_4qp - 4x_4^2q - 4x_4^2p - 2x_4p^2 - 2x_2qp - qp^2 - 32x_4^3;$
$det_{1,4,5} = -8x_4^2(2x_4 + p + 2x_2);$
$det_{1,4,6} = 2x_4p(6x_4 + p + 2x_2);$
$det_{1,4,8} = -2x_4(-4x_4 + p)(2x_4 + p + 2x_2);$
$det_{1,4,10} = 8x_4^2(2x_4 + p + 2x_2);$
$det_{1,4,12} = 2x_4p(2x_4 + p + 2x_2);$
$det_{1,4,13} = 2x_4p(-2x_4 + p + 2x_2);$
$det_{1,4,15} = -2x_4p(2x_4 + p + 2x_2);$
$det_{1,5,6} = -2x_4(-16x_4^2 + 2qx_4 + qp + 2x_2q);$
$det_{1,5,9} = -16x_4^2(2x_4 + p + 2x_2);$
$det_{1,5,12} = -2qx_4(2x_4 + p + 2x_2);$
$det_{1,5,13} = -2x_4(16x_4^2 + 2qx_4 + qp + 2x_2q);$
$det_{1,6,8,} =, 2x_4qp - 4x_4^2q + 4x_4^2p + 2x_4p^2 + 2x_2qp + qp^2 + 32x_4^3 + 4x_4x_2p - 4x_4x_2q;$
$det_{1,6,9} = 4x_4p(6x_4 + p + 2x_2);$
$det_{1,6,13} = 4x_4p(4x_4 + q);$
$det_{1,6,15} = p(4x_4x_2 + 4qx_4 + 4x_4^2 + 2x_4p + 2x_2q + qp);$
$det_{1,7,8} = -2x_4(16x_4^2 + qp);$
$det_{1,7,9} = -16x_4^2p;$
$det_{1,7,12} = -2x_4p(4x_4 + q);$
$det_{1,7,15} = -2x_4qp;$
$det_{1,8,9} = -4x_4(-4x_4 + p)(2x_4 + p + 2x_2);$
$det_{1,8,12} = -(qp + 2x_4p - 2qx_4)(2x_4 + p + 2x_2);$
$det_{1,8,13} = 2x_4qp + 4x_4^2q - 4x_4^2p - 2x_4p^2 - 2x_2qp - qp^2 + 32x_4^3 - 4x_4x_2p + 4x_4x_2q;$
$det_{1,9,12} = -4x_4p(2x_4 + p + 2x_2);$
$det_{1,9,13} = -4x_4p(-2x_4 + p + 2x_2);$
$det_{1,9,15} = 4x_4p(2x_4 + p + 2x_2);$
$det_{1,10,12} = 2qx_4(2x_4 + p + 2x_2);$
$det_{1,10,13} = 2x_4(16x_4^2 + 2qx_4 + qp + 2x_2q);$
$det_{1,11,13} = -2x_4p(-2x_4 + p + 2x_2);$
$det_{1,12,15} = p(2x_4 + q)(2x_4 + p + 2x_2);$
$det_{1,13,15} = p(4x_4x_2 + 4x_4^2 + 2x_4p + 2x_2q + qp);$
$det_{2,3,5} = -8x_4^2q;$
$det_{2,3,6} = 2qx_4(4x_4 + p);$
$det_{2,3,8} = 16x_4^2q;$
$det_{2,3,10} = 8x_4^2q;$
$det_{3,4,6} = 2x_4(-16x_4^2 + p^2 + 2x_2p + 2x_4p);$
$det_{3,4,8} = 16x_4^2(2x_4 + p + 2x_2);$
$det_{3,4,13} = 2x_4(16x_4^2 + 2x_4p + p^2 + 2x_2p);$
$det_{5,6,8} = -2qx_4(6x_4 + p + 2x_2);$
$det_{5,6,13} = -16x_4^2q;$
$det_{6,7,8} = -2qx_4(-4x_4 + p);$
$det_{6,7,9} = 2x_4p(-4x_4 + q);$
$det_{7,8,9} = -2x_4(-16x_4^2 + qp);$
$det_{8,9,10} = 2x_4(2x_4 + p + 2x_2)(-4x_4 + q);$
$det_{8,9,11} = 2x_4(-4x_4 + p)(2x_4 + p + 2x_2);$
$det_{8,9,12} = (2x_4 + p + 2x_2)(-2x_4p + qp - 2qx_4);$
$det_{8,9,13} = -4x_4x_2q - 4x_4x_2p - 2x_4qp - 4x_4^2q - 4x_4^2p - 2x_4p^2 + 2x_2qp + qp^2 + 32x_4^3;$
$det_{8,9,15} = -2x_4(2x_4 + p + 2x_2)(-4x_4 + q);$

$det_{9,10,13} = 2x_4(-16x_4^2 + 2qx_4 + qp + 2x_2q)$;
$det_{13,14,15} = -qp(2x_2 + p)$.

Proposition 7.25. *If one of the following inequalities hold*

1. $gcd(x_4, p) = k_1 > 1$;
2. $gcd(x_4, q) = k_2 > 1$;
3. $gcd(x_4, p + 2x_2) = k_3 > 1$.

then ξ is not an epimorphism.

Proof. If inequality j holds then all det_{i_1,i_2,i_3} are multiples of $k_j > 1$.

□

Proposition 7.26. *If $x_1 + x_3$ is odd, $gcd(x_1, x_3, x_4) = 1$ and $gcd(x_4, p) = gcd(x_4, q) = gcd(x_4, p + 2x_2) = 1$, then ξ is an epimorphism.*

Proof. We start by seing that if $x_1 + x_3$ is odd then p, q and $p + 2x_2$ are also odd. Compute the *gcd* between $det_{1,2,5}$ and $det_{13,14,15}$.

□

7.6. Case l=6

When computing $\phi(\pi_1(p(\mathcal{G}), id))$ we obtain only 3 distinct elements different from zero:

$$[x_1 + x_2, -x_1 - x_2, x_1 + x_3, -x_1 - x_3]id,$$

$$[x_2, x_3, -x_2, -x_3]id,$$

$$[x_1 + x_2 + x_3, -x_1, x_1, -x_1 - x_2 - x_3]id$$

Proposition 7.27. *The morphism ξ is an epimorphism if and only if*

$$det_{1,2,3} = (x_3 - x_2)(x_3 + x_2)(x_3 + x_2 + 2x_1) = \pm 1.$$

So we have 8 epimorphisms, $\xi_{(0,\pm 1,0)}$, $\xi_{(\mp 1,0,\pm 1)}$, $\xi_{(0,0,\pm 1)}$ and $\xi_{(\pm 1,\mp 1,0)}$.

7.7. Case l=7

When computing $\phi(\pi_1(p(\mathcal{G}), id))$ we obtain only 3 distinct elements different from zero:

$$[x_1 + x_2, -x_1 - x_2, -x_1 - x_3, x_3 + x_1]id,$$

$$[x_1 + x_2 + x_3, -x_1, -x_1 - x_2 - x_3, x_1]id,$$

$$[x_2, x_3, -x_3, -x_2]id.$$

Proposition 7.28. *The morphism ξ is an epimorphism if and only if*

$$det_{1,2,3} = (x_3 - x_2)(x_3 + x_2)(x_3 + x_2 + 2x_1) = \pm 1.$$

So we have 8 epimorphisms, $\xi_{(0,\pm 1,0)}$, $\xi_{(\mp 1,0,\pm 1)}$, $\xi_{(0,0,\pm 1)}$ and $\xi_{(\pm 1,\mp 1,0)}$.

8. Classes of epimorphisms

In this section we will see which among the previously founded epimorphisms are actually different modulus an automorphism of $W(\widetilde{A}_{n-1})$, $n > 2$. We will proceed by analyzing each nontrivial case introduced in section 7. Note that case 2 has already been eliminated.

Excepting the case $l = 1$, for all other cases we will compute the image of $\xi(\sigma_i)$ by all graph automorphisms of $W(\widetilde{A_{n-1}})$).

The sequence of graph automorphisms will be $id, \gamma, \rho, \rho\gamma, \rho^2, \rho^2\gamma, \rho^3, \rho^3\gamma$, where ρ and γ are, respectively, the rotation and the reflection, introduced in section 6. Note that this group is the dihedral group of order 8.

8.1. Case $l = 1$

Proposition 8.1. *Let $(y_1, p_1), (y_2, p_2) \in \mathbb{Z}^2$ be different and $\xi_{(y_1,p_1)}$, $\xi_{(y_2,p_2)}$ be two epimorphisms from $A(\widetilde{A}_{n-1})$ to $W(\widetilde{A}_{n-1})$. It does not exist an automorphism ϕ of $W(\widetilde{A}_{n-1})$ such that*

$$\xi_{y_2,p_2} = \xi_{y_1,p_1} \circ \phi.$$

Proof. Suppose that exist a ϕ in the above conditions. We may assume that its restriction to S_n is, modulo graph conjugation in S_n, the identity. Otherwise we had a new class of epimorphism of $A(\widetilde{A}_{n-1})$ to S_n which different from ξ_1. We have that, for

$$\xi(\sigma_i) = [\underbrace{y, \ldots, y}_{i-1}, -(n-2)y - x_i, x_i, y, \ldots, y](i, i+1)$$

$$\phi(\xi(\sigma_i)) = [u_1, \ldots, u_n](i, i+1).$$

So remains to see how does it behaves when restricted to $\xi(Ker(p \circ \xi))$. Let $g = [Q_1(y,p), \ldots, Q_n(y,p)]id$, in this case being ϕ an inner automorphism, a simple computation, shows that

$$\phi(g) = [Q_{s(1)}(y,p), \ldots, Q_{s(n)}(y,p)]id$$

for some permutation $s \in S_n$. So modulo inner by graph automorphisms of $A(\widetilde{A}_{n-1})$ we must have $(y_1, p_1) = (y_2, p_2)$.

□

Now we will see how the inner by graph automorphisms of $W(\widetilde{A}_{n-1})$ acts on $\xi_{y,p}$, for $gcd(y,p) = 1$.

Lemma 8.2. *Let $y, p \in \mathbb{Z}$ and $gcd(y,p) = 1$. Then we have:*

$$\rho((i,n)) = [1, \underbrace{0, \ldots, 0}_{i-1}, -1, 0, \ldots, 0](1, i+1) = w_1 w_{i+1}^{-1}(1, i+1), \text{ for } 1 \le i < n,$$

$$\rho((1,i)) = (2, i+1) = s_1(1, i+1)s_1, \text{ for } 1 < i \le n-1,$$

Proof. We can write the permutation (i,n) in terms of the generators s_i of $W(\widetilde{A}_{n-1})$. So $(i,n) = s_i \cdots s_{n-1}s_{n-2} \cdots s_i$, hence, by definition, $\rho((i,n)) = s_{i+1} \cdots s_n s_{n-1} \cdots s_{i+1}$. We simplify to obtain $\rho((i,n)) = [1, \underbrace{0, \ldots, 0}_{i-1}, -1, 0, \ldots, 0]s_{i+1} \cdots s_{n-1}(1,n)s_{n-1} \cdots s_{i+1} = [1, \underbrace{0, \ldots, 0}_{i-1}, -1, 0, \ldots, 0](1, i)$.
The proof of the second equality is very similar. We have $(1,i) = s_1 \cdots s_{i-1}s_{i-2} \cdots s_1$, $i \le n$, and, by definition $\rho((1,i)) = s_2 \cdots s_i s_{i-1} \cdots s_2 = (2, i+1)$, $i \le n-1$.

□

Proposition 8.3. *Let $n > 2$, $1 \leq i \leq n$, $y, p \in \mathbb{Z}$ and $gcd(y, p) = 1$. The rotation ρ acts on $\xi_{y,p}$ in the following way:*

$$\rho(\xi_{y,p}(\sigma_i)) = \xi_{y,p}(\sigma_{i+1}) \text{ for } 1 \leq i \leq n-2;$$

$$\rho(\xi_{y,p}(\sigma_{n-1})) = w_1^{1-p}\xi_{y,p}(\sigma_n);$$

$$\rho(\xi_{y,p}(\sigma_n)) = w_1^{1-p}w_2^{p-1}\xi_{y,p}(\sigma_1).$$

Proof. This is a simple computation:

Suppose that $i \neq n-1, n$.

$$\xi_{y,p}(\sigma_i) = [\underbrace{y, \ldots, y}_{i-1}, -(n-2)y, 0, y, \ldots, y](i, i+1) =$$
$$= (i, n)[\underbrace{y, \ldots, y}_{i}, 0, y, \ldots, y, -(n-2)y](i, n)(i, i+1) =$$
$$= (i, n) \prod_{\substack{k \in \{1, \ldots, n-1\} \\ k \neq i+1}} w_k^y \, (i, n)s_i,$$

so, by proposition 6.5, we have

$$\rho(\xi_{y,p}(\sigma_i)) = \rho((i, n))w_1^{-(n-2)y} \prod_{\substack{k \in \{1, \ldots, n-2\} \\ k \neq i+1}} w_{k+1}^y \, \rho((i, n))s_{i+1} =$$
$$= \rho((i, n))w_1^{-(n-2)y} \prod_{\substack{k \in \{2, \ldots, n-1\} \\ k \neq i+2}} w_k^y \, \rho((i, n))s_{i+1} =$$
$$= \rho((i, n))(1, i+1)\xi_{y,p}(\sigma_{i+1})s_{i+1}(1, i+1)\rho((i, n))s_{i+1} =$$

$$= w_1w_{i+1}^{-1}(1, i+1)(1, i+1)\xi_{y,p}(\sigma_{i+1})s_{i+1}(1, i+1)w_1w_{i+1}^{-1}(1, i+1)s_{i+1} =$$
$$= w_1w_{i+1}^{-1}\xi_{y,p}(\sigma_{i+1})s_{i+1}w_{i+1}w_1^{-1}(1, i+1)(1, i+1)s_{i+1} =$$
$$= w_1w_{i+1}^{-1}\xi_{y,p}(\sigma_{i+1})w_{i+2}w_1^{-1} =$$
$$= w_{i+1}^{-1}\xi_{y,p}(\sigma_{i+1})w_{i+2} =$$
$$= w_{i+1}^{-1}w_{i+1}\xi_{y,p}(\sigma_{i+1}) =$$
$$= \xi_{y,p}(\sigma_{i+1}).$$

Suppose that $i = n-1$.

$$\xi_{y,p}(\sigma_{n-1}) = [y, \ldots, y, -(n-2)y, 0](n-1, n) =$$
$$= (n-1, n)[y, \ldots, y, 0, -(n-2)y] =$$
$$= s_{n-1} \prod_{k \in \{1, \ldots, n-2\}} w_k^y,$$

so, by proposition 6.5, we have

$$\rho(\xi_{y,p}(\sigma_{n-1})) = \rho(s_{n-1}) w_1^{-(n-2)y} \prod_{k \in \{1,...,n-2\}} w_{k+1}^y =$$

$$= \rho(s_{n-1}) w_1^{-(n-2)y} \prod_{k \in \{2,...,n-1\}} w_k^y = w_1(1,n) w_1^{-(n-2)y} \prod_{k \in \{2,...,n-1\}} w_k^y =$$

$$= w_1(1,n)(1,n)\xi_{y,0}(\sigma_n) = w_1^{1-p}\xi_{y,p}(\sigma_n).$$

Suppose that $i = n$.

$$\xi_{y,p}(\sigma_n) = [p, y, \ldots, y, -(n-2)y - p](1,n) = w_1^p \prod_{k\in\{2,\ldots,n-1\}} w_k^y \,(1,n),$$

so, by proposition 6.5, we have

$$\rho(\xi_{y,p}(\sigma_n)) = w_2^p w_1^{-y-p} \prod_{k\in\{1,\ldots,n-2\}} w_1^{-y} w_{k+1}^y \,\rho((1,n)) =$$

$$\rho(\xi_{y,p}(\sigma_n)) = w_2^p w_1^{-(n-2)y-p} \prod_{k\in\{2,\ldots,n-1\}} w_k^y \,\rho((1,n)) =$$

$$= [-(n-2)y - p, p, y, \ldots, y] s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 = \xi_{y,p}(\sigma_1) s_1 \rho((1,n)) = \xi_{y,p}(\sigma_1) s_1 w_1 w_2^{-1} s_1 =$$

$$= \xi_{y,p}(\sigma_1) w_2 w_1^{-1} = w_1^{1-p} w_2^{p-1} \xi_{y,p}(\sigma_1).$$

□

Proposition 8.4. *Let $k > 1$, $n > 2$, $1 \leq i \leq n$, $y, p \in \mathbb{Z}$ and $gcd(y,p) = 1$ We have*

$$\rho^k(\xi_{y,p}(\sigma_i)) = \xi_{y,p}(\sigma_{i+k \ mod \ n}) \text{ for } i + k - 1 \ mod \ n \neq 0, n-1 \text{ and } i \neq n;$$

$$\rho^k(\xi_{y,p}(\sigma_i)) = w_1^{1-p}\xi_{y,p}(\sigma_n) \text{ for } i + k - 1 \ mod \ n = n - 1; \text{ and } i \neq n$$

$$\rho^k(\xi_{y,p}(\sigma_i)) = w_2^{1-p} w_1^{p-1} \xi_{y,p}(\sigma_1) \text{ for } i + k - 1 \ mod \ n = 0 \text{ and } i \neq n;$$

$$\rho^k(\xi_{y,p}(\sigma_n)) = w_k w_{k+1}^{-1} \xi_{y,p}(\sigma_k) \text{ for } k - 1 \ mod \ n \neq n - 1;$$

$$\rho^{n-1}(\xi_{y,p}(\sigma_n)) = w_{n-1}\xi_{y,p}(\sigma_{n-1}).$$

Proof. This proof is a direct use of Proposition 6.5.

□

Remark 8.5. Note that, $\rho^n(\xi_{y,p}(\sigma_i)) = \xi_{y,p}(\sigma_i)$ for $1 \leq i \leq n$.

Proposition 8.6. *In the previous conditions we have:*

$$\gamma(\xi_{y,p}(\sigma_1)) = s_1\xi_{-y,p}(\sigma_1)s_1;$$

$$\gamma(\xi_{y,p}(\sigma_2)) = w_1^{-p}(1,n)\xi_{-y,p}(\sigma_n)s_n;$$

$$\gamma(\xi_{y,p}(\sigma_n)) = w_2^{1-p}w_3^{p-1}s_2\xi_{-y,p}(\sigma_2)s_2.$$

$$\gamma(\xi_{y,p}(\sigma_i)) = s_{n-i+2}\xi_{-y,p}(\sigma_{n-i+2})s_{n-i+2} \text{ for } i \neq 1,2,n;$$

Proof. We start by the first case.

$$\gamma(\xi_{y,p}(\sigma_1)) = \gamma([-(n-2)y, 0, y, \dots, y]s_1) = \gamma(w_1^{-(n-2)y} \prod_{k\in\{3,\dots,n-1\}} w_k^y)s_1 =$$

$$= w_2^{(n-2)y}w_3^{-(n-2)y}w_3^y \prod_{k\in\{4,\dots,n-1\}} w_3^y w_{n+3-k}^{-y}\, s_1 =$$

$$= w_2^{(n-2)y}w_3^{-(n-2)y}w_3^{(n-3)y} \prod_{k\in\{4,\dots,n-1\}} w_{n+3-k}^{-y}\, s_1 =$$

$$= w_2^{(n-2)y} \prod_{k\in\{3,\dots,n-1\}} w_k^{-y}\, s_1 = s_1\xi_{-y,p}(\sigma_1)s_1.$$

For the second case we have:

$$\gamma(\xi_{y,p}(\sigma_2)) = \gamma([y, -(n-2)y, 0, y, \dots, y]s_2) = \gamma(w_2^{-(n-2)y}w_1^y \prod_{k\in\{4,\dots,n-1\}} w_k^y)s_n =$$

$$= w_1^{(n-2)y}w_3^{-(n-2)y}w_2^{-y}w_3^y \prod_{k\in\{4,\dots,n-1\}} w_3^y w_{n+3-k}^{-y} s_n =$$

$$= w_1^{(n-2)y}w_3^{-(n-2)y}w_2^{-y}w_3^{-(n-3)y} \prod_{k\in\{4,\dots,n-1\}} w_{n+3-k}^{-y} s_n =$$

$$= w_1^{(n-2)y}w_3^{-y}w_2^{-y} \prod_{k\in\{4,\dots,n-1\}} w_k^{-y} s_n =$$

$$= w_1^p w_1^{(n-2)y-p} \prod_{k\in\{2,\dots,n-1\}} w_k^{-y} s_n = w_1^{-p}(1,n)\xi_{-y,p}(\sigma_n)s_n$$

For the third case we have:

$$\gamma(\xi_{y,p}(\sigma_n)) = \gamma([p, y, \dots, y, -(n-2)y-p]w_1^{-1}s_n) = \gamma(w_1^p w_2^y w_3^y \prod_{k\in\{4,\dots,n-1\}} w_k^y)w_2w_3^{-1}s_2 =$$

$$= w_2^{-p}w_3^p w_3^y w_1^{-y} w_3^y \prod_{k\in\{4,\dots,n-1\}} w_3^y w_{n+3-k}^{-y}\, w_2w_3^{-1}s_2 =$$

$$= w_2^{-p} w_3^{(n-2)y+p} w_1^{-y} \prod_{k\in\{4,\dots,n-1\}} w_{n+3-k}^{-y}\, w_2 w_3^{-1} s_2 =$$

$$= w_2^{-p} w_3^{(n-2)y+p} \prod_{k\in\{1,\dots,n-1\}} w_k^{-y}\, w_2 w_3^{-1} s_2 = w_2^{1-p} w_3^{p-1} s_2 \xi_{-y,p}(\sigma_2) s_2$$

Although the case σ_3 as the same result that the general case, the proof is slightly different:

$$\gamma(\xi_{y,p}(\sigma_3)) = \gamma([y,y,-(n-2)y,0,y,\dots,y]s_3) = \gamma(w_1^y w_2^y w_3^{-(n-2)y} \prod_{k\in\{5,\dots,n-1\}} w_k^y) s_{n-1} =$$

$$= w_2^{-y} w_3^y w_1^{-y} w_3^y w_3^{-(n-2)y} \prod_{k\in\{5,\dots,n-1\}} w_3^y w_{n+3-k}^{-y} s_{n-1} =$$

$$= w_2^{-y} w_1^{-y} w_3^{-y} \prod_{k\in\{5,\dots,n-1\}} w_{n+3-k}^{-y} s_{n-1} = s_{n-1}\xi_{-y,p}(\sigma_{n-1}) s_{n-1}$$

Finally for the last, and general, case. Suppose that $3 < i < n$.

$$\gamma(\xi_{y,p}(\sigma_i)) = \gamma([\underbrace{y,\dots,y}_{i-1},-(n-2)y,0,y,\dots,y]s_i) = \gamma(w_1^y w_2^y w_3^y w_i^{-(n-2)y} \prod_{k\in\{4,\dots,i-1,i+2,\dots,n-1\}} w_k^y) s_{n-i+2} =$$

$$= w_3^y w_2^{-y} w_1^{-y} w_3^y w_3^y w_3^{-(n-2)y} w_{n+3-i}^{(n-2)y} \prod_{k\in\{4,\dots,i-1,i+2,\dots,n-1\}} w_3^y w_{n+3-k}^{-y} s_{n-i+2} =$$

$$= w_3^y w_2^{-y} w_1^{-y} w_3^y w_3^y w_3^{-(n-2)y} w_{n+3-i}^{(n-2)y} w_3^{(n-6)y} \prod_{k\in\{4,\dots,i-1,i+2,\dots,n-1\}} w_{n+3-k}^{-y} s_{n-i+2} =$$

$$= w_2^{-y} w_1^{-y} w_3^{-(n-5)y} w_{n+3-i}^{(n-2)y} w_3^{(n-6)y} \prod_{k\in\{4,\dots,i-1,i+2,\dots,n-1\}} w_{n+3-k}^{-y} s_{n-i+2} =$$

$$= w_2^{-y} w_1^{-y} w_3^{-y} w_{n+3-i}^{(n-2)y} \prod_{k\in\{4,\dots,i-1,i+2,\dots,n-1\}} w_{n+3-k}^{-y} s_{n-i+2} = s_{n-i+2}\xi_{-y,p}(\sigma_{n-i+2}) s_{n-i+2}$$

□

Proposition 8.7. *Let $k > 0$, $n > 2$, $1 \le i \le n$, $y,p \in \mathbb{Z}$ and $gcd(y,p) = 1$ We have*

1. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = w_1^{-p}(1,n)\xi_{-y,p}(\sigma_n)s_n$ *for* $i+k-1 \mod n = 1$ *and* $i \neq n$;
2. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \xi_{y,p}(\sigma_{n+3-(i+k \mod n)})$ *for* $i+k-1 \mod n \neq 0,1,n-1$ *and* $i \neq n$;
3. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = w_3^{p-1} s_2 \xi_{-y,p}(\sigma_2) s_2$ *for* $i+k-1 \mod n = n-1$; *and* $i \neq n$
4. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = w_1^{1-p} w_2^{p-1} s_1 \xi_{-y,p}(\sigma_1) s_1$ *for* $i+k-1 \mod n = 0$ *and* $i \neq n$;
5. $\gamma\rho^2(\xi_{y,p}(\sigma_n)) = w_2^{-1} w_1^{-p}(1,n)\xi_{-y,p}(\sigma_n)s_n$ *for* $k-1 \mod n = 1$;
6. $\gamma\rho^3(\xi_{y,p}(\sigma_n)) = w_4 s_{n-1}\xi_{-y,p}(\sigma_{n-1})s_{n-1}$ *for* $k-1 \mod n = 2$;
7. $\gamma\rho^k(\xi_{y,p}(\sigma_n)) = w_{k+1} w_k^{-1} s_{n+3-k}\xi_{-y,p}(\sigma_{n+3-k})s_{n+3-k}$ *for* $k-1 \mod n \neq 1,2,n-1$;
8. $\gamma\rho^{n-1}(\xi_{y,p}(\sigma_n)) = w_3 s_3 \xi_{-y,p}(\sigma_3) s_3$.

Proof. We will follow the cases of Proposition 8.4 one by one:

1. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \gamma(\xi_{y,p}(\sigma_{i+k \bmod n}))$ for $i+k-1 \bmod n \neq 0, n-1$ and $i \neq n$;
   If $i+k \bmod n = 2$ then
$$\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \gamma(\xi_{y,p}(\sigma_2)) = w_1^{-p}(1,n)\xi_{-y,p}(\sigma_n)s_n.$$
   If $i+k \bmod n = n$ then
$$\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \gamma(\xi_{y,p}(\sigma_n)) = w_2^{1-p}w_3^{p-1}s_2\xi_{-y,p}(\sigma_2)s_2.$$
   If $i+k \bmod n \neq 1, 2, n$ then
$$\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \gamma(\xi_{y,p}(\sigma_{i+k \bmod n})) = \xi_{y,p}(\sigma_{n+3-(i+k \bmod n)}).$$
2. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \gamma(w_1^{1-p}\xi_{y,p}(\sigma_n)) =$
   $= w_3^{p-1}s_2\xi_{-y,p}(\sigma_2)s_2$ for $i+k-1 \bmod n = n-1$; and $i \neq n$
3. $\gamma\rho^k(\xi_{y,p}(\sigma_i)) = \gamma(w_2^{1-p}w_1^{p-1}\xi_{y,p}(\sigma_1)) =$
   $= w_1^{1-p}w_2^{p-1}s_1\xi_{-y,p}(\sigma_1)s_1$ for $i+k-1 \bmod n = 0$ and $i \neq n$;
4. $\gamma\rho^k(\xi_{y,p}(\sigma_n)) = \gamma(w_k w_{k+1}^{-1}\xi_{y,p}(\sigma_k))$ for $k-1 \bmod n \neq n-1$;
   If $k-1 \bmod n = 1$ then
$$\gamma\rho^2(\xi_{y,p}(\sigma_n)) = \gamma(w_2w_3^{-1}\xi_{y,p}(\sigma_2)) = w_2^{-1}w_1^{-p}(1,n)\xi_{-y,p}(\sigma_n)s_n.$$
   If $k-1 \bmod n = 2$ then
$$\gamma\rho^3(\xi_{y,p}(\sigma_n)) = \gamma(w_3w_4^{-1}\xi_{y,p}(\sigma_3)) = w_4s_{n-1}\xi_{-y,p}(\sigma_{n-1})s_{n-1}.$$
   If $k-1 \bmod n \neq 1, 2, n-1, n$ then
$$\gamma\rho^k(\xi_{y,p}(\sigma_n)) = \gamma(w_kw_{k+1}^{-1}\xi_{y,p}(\sigma_k)) = w_{k+1}w_k^{-1}s_{n+3-k}\xi_{-y,p}(\sigma_{n+3-k})s_{n+3-k}.$$
5. $\gamma\rho^{n-1}(\xi_{y,p}(\sigma_n)) = \gamma(w_{n-1}\xi_{y,p}(\sigma_{n-1})) = w_3s_3\xi_{-y,p}(\sigma_3)s_3.$

□

We will deal now with the standard epimorphism μ and $\xi_{0,-1}$. Notice that $\rho(\mu(\sigma_i)) = \rho(\xi_{0,-1}(\sigma_i))$ for $i \neq n$.

Lemma 8.8. *Let $s'_n = \xi_{0,-1}(\sigma_n)$. Then $\rho^k(s'_n) \neq \rho^k(s_n)$.*

Proof. We can suppose that $k < n$, because $\rho^n = id$. So

Using Proposition 8.4 we have, for $k < n-1$:
$$\rho^k(s'_n) = \rho^k(\xi_{0,-1}(\sigma_n)) = w_kw_{k+1}^{-1}\xi_{0,-1}(\sigma_k) =$$
$$= w_kw_{k+1}^{-1}\xi_{0,1}(\sigma_k) \neq \xi_{0,1}(\sigma_k) = \mu(\sigma_k) = \rho^k(\xi_{0,1}(\sigma_n)).$$

□

Lemma 8.9. *Let $n > 2$, $k > 1$, $s'_n = \xi_{0,-1}(\sigma_n)$. Then $\gamma(s'_n) \neq \gamma(s_n)$.*

Proof. By Proposition 8.6 we have:

$$\gamma(s'_n) = \gamma(\xi_{0,-1}(\sigma_n)) = w_2^2 w_3^{-2} s_2 \xi_{0,-1}(\sigma_2) s_2 =$$
$$= w_2^2 w_3^{-2} s_2 \xi_{0,1}(\sigma_2) s_2 = w_2^2 w_3^{-2} \gamma(\xi_{0,1}(\sigma_n)) =\neq \gamma(\xi_{0,1}(\sigma_n)) = \gamma(s_n).$$

□

Lemma 8.10. *Let $n > 2$, $k > 1$, $s'_n = \xi_{0,-1}(\sigma_n)$. Then $\gamma\rho^k(s'_n) \neq \gamma\rho^k(s_n)$.*

Proof. We can suppose, again, that $k < n$. So

Using Proposition 8.4 we have, for $k < n-1$:

$$\gamma\rho^k(s'_n) = \gamma\rho^k(\xi_{0,-1}(\sigma_n)) = \gamma(w_k w_{k+1}^{-1} \xi_{0,-1}(\sigma_k)) =$$

$$= \gamma(w_k w_{k+1}^{-1}) \gamma(\xi_{0,1}(\sigma_k)) = \gamma(w_k w_{k+1}^{-1}) \gamma(s_k)$$

If $k = 1$ then

$$\gamma\rho(s'_n) = \gamma(w_1 w_2^{-1}) \gamma(s_1) = w_1 w_2^{-1} s_1 \neq s_1 = \gamma(s_1) = \gamma\rho(s_n).$$

If $k = 2$ then

$$\gamma\rho^2(s'_n) = \gamma(w_2 w_3^{-1}) \gamma(s_2) = w_1^{-1} s_n \neq s_n = \gamma(s_2) = \gamma\rho^2(s_n).$$

If $k = 3$ then

$$\gamma\rho^3(s'_n) = \gamma(w_3 w_4^{-1}) \gamma(s_3) = w_{n-1} s_{n-1} \neq s_{n-1} = \gamma(s_3) = \gamma\rho^3(s_n).$$

For $k > 3$ then

$$\gamma\rho^k(s'_n) = \gamma(w_k w_{k+1}^{-1}) \gamma(s_k) = w_{n+2-k} w_{n+3-k}^{-1} s_{n+3-k} \neq s_{n+3-k} = \gamma(s_k) = \gamma\rho^k(s_n).$$

□

Proposition 8.11. *Let $n > 2$, then the morphisms $\mu = \xi_{0,1}$ and $\xi_{0,-1}$ are different up to automorphisms of $W(\widetilde{A}_{n-1})$.*

Proof. Lemmas 8.8, 8.9 and 8.10 show to us that there are no automorphism ψ of $W(\widetilde{A}_{n-1})$ such that

$$\psi(\xi_{0,-1}(\sigma_n)) = \xi_{0,1}(\sigma_n) = \mu(\sigma_n).$$

□

Finally we state:

Proposition 8.12. *If $(y, p) \neq (y', p')$ then the classes, up to automorphisms of $W(\widetilde{A}_{n-1})$, $\xi_{y,p}$ and $\xi_{y',p'}$ are different.*

Proof. For a fixed pair (y, p), using the results of Propositions 8.3, 8.4 and 8.6, we conclude that each automorphism, ψ, of $W(\widetilde{A}_{n-1})$ sends $(\xi_{y,p}(\sigma_1), \ldots, \xi_{y,p}(\sigma_n))$ to a different list $(\psi(\xi_{y,p}(\sigma_1)), \ldots, \psi(\xi_{y,p}(\sigma_n)))$. The same occurs if we change the pair (y, p). We never obtain a repeated list.

□

8.2. Case $l = 3$

For the epimorphism $\xi_{(1,0,0)}$ we have:

	$\xi_{(1,0,0)}(\sigma_1)$	$\xi_{(1,0,0)}(\sigma_2)$	$\xi_{(1,0,0)}(\sigma_3)$	$\xi_{(1,0,0)}(\sigma_4)$
id	[[0,-1,1,0],[[1,2,3,4]]]	[[-1,0,1,0],[[1,3,4,2]]]	[[0,-1,1,0],[[1,2,3,4]]]	[[0,0,1,-1],[[1,2,4,3]]]
γ	[[1,1,0,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,2,4 ,3]]]	[[1,1,0,-2],[[1,4,3,2]]]	[[0,1,0,-1],[[1,3,4,2]]]
ρ	[[1,-1,-1,1],[[1,2,3,4]]]	[[1,-1,-1,1],[[1,3 ,2,4]]]	[[1,-1,-1,1],[[1,2,3,4]]]	[[0,0,0,0],[[1,4,2,3]]]
$\rho\gamma$	[[-1,1,1,-1],[[1,4,3,2]]]	[[-1,1,1,-1],[[1,4 ,2,3]]]	[[-1,1,1,-1],[[1,4,3,2]]]	[[0,0,0,0],[[1,3,2,4]]]
ρ^2	[[2,0,-1,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,2 ,4,3]]]	[[2,0,-1,-1],[[1,2,3,4]]]	[[1,0,-1,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,-1,1,0],[[1,4,3,2]]]	[[0,-1,0,1],[[1,3,4 ,2]]]	[[0,-1,1,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,2,4,3]]]
ρ^3	[[0,1,0,-1],[[1,2,3,4]]]	[[0,2,0,-2],[[1,4,2 ,3]]]	[[0,1,0,-1],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,0,-1,0],[[1,4,3,2]]]	[[2,0,-2,0],[[1,3,2 ,4]]]	[[1,0,-1,0],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(-1,0,0)}$ we have:

	$\xi_{(-1,0,0)}(\sigma_1)$	$\xi_{(-1,0,0)}(\sigma_2)$	$\xi_{(-1,0,0)}(\sigma_3)$	$\xi_{(-1,0,0)}(\sigma_4)$
id	[[0,1,-1,0],[[1,2,3,4]]]	[[1,0,-1,0],[[1,3,4,2]]]	[[0,1,-1,0],[[1,2,3,4]]]	[[0,0,-1,1],[[1,2,4,3]]]
γ	[[-1,1,0,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[-1,1,0,0],[[1,4,3,2]]]	[[0,1,-2,1],[[1,3,4,2]]]
ρ	[[1,-1,1,-1],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,3 ,2,4]]]	[[1,-1,1,-1],[[1,2,3,4]]]	[[2,0,0,-2],[[1,4,2,3]]]
$\rho\gamma$	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,4 ,2,3]]]	[[1,-1,1,-1],[[1,4,3,2]]]	[[2,0,0,-2],[[1,3,2,4]]]
ρ^2	[[0,0,-1,1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,2,4 ,3]]]	[[0,0,-1,1],[[1,2,3,4]]]	[[-1,2,-1,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,1,-1,0],[[1,4,3,2]]]	[[0,1,0,-1],[[1,3,4 ,2]]]	[[0,1,-1,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,2,4,3]]]
ρ^3	[[2,-1,0,-1],[[1,2,3,4]]]	[[0,0,0,0],[[1,4,2 ,3]]]	[[2,-1,0,-1],[[1,2,3,4]]]	[[1,-1,1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,0,1,-2],[[1,4,3,2]]]	[[0,0,0,0],[[1,3,2, 4]]]	[[1,0,1,-2],[[1,4,3,2]]]	[[1,-1,1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(1,-1,0)}$ we have:

	$\xi_{(1,-1,0)}(\sigma_1)$	$\xi_{(1,-1,0)}(\sigma_2)$	$\xi_{(1,-1,0)}(\sigma_3)$	$\xi_{(1,-1,0)}(\sigma_4)$
id	[[0,0,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,3,4,2]]]	[[0,0,1,-1],[[1,2,3,4]]]	[[0,1,0,-1],[[1,2,4,3]]]
γ	[[0,1,1,-2],[[1,4,3,2]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[0,1,1,-2],[[1,4,3,2]]]	[[-1,1,0,0],[[1,3,4,2]]]
ρ	[[0,-1,0,1],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,-1,0,1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[-1,0,1,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[-1,0,1,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3,2,4]]]
ρ^2	[[2,-1,-1,0],[[1,2,3,4]]]	[[2,-1,0,-1],[[1,2 ,4,3]]]	[[2,-1,-1,0],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,4,2]]]
$\rho^2\gamma$	[[1,-1,0,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3,4 ,2]]]	[[1,-1,0,0],[[1,4,3,2]]]	[[1,0,-1,0],[[1,2,4,3]]]
ρ^3	[[1,1,-1,-1],[[1,2,3,4]]]	[[0,2,-1,-1],[[1,4 ,2,3]]]	[[1,1,-1,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,1,-1,-1],[[1,4,3,2]]]	[[1,1,-2,0],[[1,3, 2,4]]]	[[1,1,-1,-1],[[1,4,3,2]]]	[[1,1,0,-2],[[1,4,2,3]]]

For the epimorphism $\xi_{(-1,1,0)}$ we have:

	$\xi_{(-1,1,0)}(\sigma_1)$	$\xi_{(-1,1,0)}(\sigma_2)$	$\xi_{(-1,1,0)}(\sigma_3)$	$\xi_{(-1,1,0)}(\sigma_4)$
id	[[0,0,-1,1],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,4,2]]]	[[0,0,-1,1],[[1,2,3,4]]]	[[0,-1,0,1],[[1,2,4,3]]]
γ	[[0,1,-1,0],[[1,4,3,2]]]	[[1,0,-1,0],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,4,3,2]]]	[[1,1,-2,0],[[1,3,4,2]]]
ρ	[[2,-1,0,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,0,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[1,0,1,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[1,0,1,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,3,2,4]]]
ρ^2	[[0,1,-1,0],[[1,2,3,4]]]	[[0,1,0,-1],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,2,3,4]]]	[[0,2,-1,-1],[[1,3,4,2]]]
$\rho^2\gamma$	[[-1,1,0,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,3,4 ,2]]]	[[-1,1,0,0],[[1,4,3,2]]]	[[-1,0,1,0],[[1,2,4,3]]]
ρ^3	[[1,-1,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,4, 2,3]]]	[[1,-1,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3, 2,4]]]	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,-1,0,0],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,0,1)}$ we have:

	$\xi_{(0,0,1)}(\sigma_1)$	$\xi_{(0,0,1)}(\sigma_2)$	$\xi_{(0,0,1)}(\sigma_3)$	$\xi_{(0,0,1)}(\sigma_4)$
id	[[1,-1,0,0],[[1,2,3,4]]]	[[0,-1,0,1],[[1,3,4,2]]]	[[1,-1,0,0],[[1,2,3,4]]]	[[1,-1,0,0],[[1,2,4,3]]]
γ	[[1,0,0,-1],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,2, 4,3]]]	[[1,0,0,-1],[[1,4,3,2]]]	[[1,0,-1,0],[[1,3,4,2]]]
ρ	[[1,0,-1,0],[[1,2,3,4]]]	[[2,0,-2,0],[[1,3,2 ,4]]]	[[1,0,-1,0],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[0,1,0,-1],[[1,4,3,2]]]	[[0,2,0,-2],[[1,4,2 ,3]]]	[[0,1,0,-1],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,3,2,4]]]
ρ^2	[[1,0,0,-1],[[1,2,3,4]]]	[[1,1,0,-2],[[1,2,4 ,3]]]	[[1,0,0,-1],[[1,2,3,4]]]	[[0,1,0,-1],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,0,1,-1],[[1,4,3,2]]]	[[-1,0,1,0],[[1,3,4 ,2]]]	[[0,0,1,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,2,4,3]]]
ρ^3	[[0,0,0,0],[[1,2,3,4]]]	[[-1,1,1,-1],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,2,3,4]]]	[[0,0,0,0],[[1,3,2,4]]]
$\rho^3\gamma$	[[0,0,0,0],[[1,4,3,2]]]	[[1,-1,-1,1],[[1,3,2 ,4]]]	[[0,0,0,0],[[1,4,3,2]]]	[[0,0,0,0],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,0,-1)}$ we have:

	$\xi_{(0,0,-1)}(\sigma_1)$	$\xi_{(0,0,-1)}(\sigma_2)$	$\xi_{(0,0,-1)}(\sigma_3)$	$\xi_{(0,0,-1)}(\sigma_4)$
id	[[-1,1,0,0],[[1,2,3,4]]]	[[0,1,0,-1],[[1,3,4,2]]]	[[-1,1,0,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,2,4,3]]]
γ	[[-1,2,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,2, 4,3]]]	[[-1,2,0,-1],[[1,4,3,2]]]	[[-1,2,-1,0],[[1,3,4,2]]]
ρ	[[1,-2,1,0],[[1,2,3,4]]]	[[0,0,0,0],[[1,3,2, 4]]]	[[1,-2,1,0],[[1,2,3,4]]]	[[1,-1,1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[0,-1,2,-1],[[1,4,3,2]]]	[[0,0,0,0],[[1,4,2 ,3]]]	[[0,-1,2,-1],[[1,4,3,2]]]	[[1,-1,1,-1],[[1,3,2,4]]]
ρ^2	[[1,0,-2,1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[1,0,-2,1],[[1,2,3,4]]]	[[0,1,-2,1],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,0,-1,1],[[1,4,3,2]]]	[[1,0,-1,0],[[1,3,4 ,2]]]	[[0,0,-1,1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,2,4,3]]]
ρ^3	[[2,0,0,-2],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,4, 2,3]]]	[[2,0,0,-2],[[1,2,3,4]]]	[[2,0,0,-2],[[1,3,2,4]]]
$\rho^3\gamma$	[[2,0,0,-2],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,3, 2,4]]]	[[2,0,0,-2],[[1,4,3,2]]]	[[2,0,0,-2],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,1,-1)}$ we have:

	$\xi_{(0,1,-1)}(\sigma_1)$	$\xi_{(0,1,-1)}(\sigma_2)$	$\xi_{(0,1,-1)}(\sigma_3)$	$\xi_{(0,1,-1)}(\sigma_4)$
id	[[1,0,0,-1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,4,2]]]	[[1,0,0,-1],[[1,2,3,4]]]	[[1,0,-1,0],[[1,2,4,3]]]
γ	[[0,0,1,-1],[[1,4,3,2]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[0,0,1,-1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,4,2]]]
ρ	[[0,0,0,0],[[1,2,3,4]]]	[[1,1,-2,0],[[1,3,2, 4]]]	[[0,0,0,0],[[1,2,3,4]]]	[[1,1,0,-2],[[1,4,2,3]]]
$\rho\gamma$	[[0,0,0,0],[[1,4,3,2]]]	[[0,2,-1,-1],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,3,2,4]]]
ρ^2	[[1,-1,0,0],[[1,2,3,4]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[1,-1,0,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[1,0,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,3,4 ,2]]]	[[1,0,0,-1],[[1,4,3,2]]]	[[0,1,0,-1],[[1,2,4,3]]]
ρ^3	[[1,0,-1,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[1,0,-1,0],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,2,4]]]
$\rho^3\gamma$	[[0,1,0,-1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,1,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,-1,1)}$ we have:

	$\xi_{(0,-1,1)}(\sigma_1)$	$\xi_{(0,-1,1)}(\sigma_2)$	$\xi_{(0,-1,1)}(\sigma_3)$	$\xi_{(0,-1,1)}(\sigma_4)$
id	[[1,0,0,-1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,4,2]]]	[[1,0,0,-1],[[1,2,3,4]]]	[[1,0,-1,0],[[1,2,4,3]]]
γ	[[0,0,1,-1],[[1,4,3,2]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[0,0,1,-1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,4,2]]]
ρ	[[0,0,0,0],[[1,2,3,4]]]	[[1,1,-2,0],[[1,3,2, 4]]]	[[0,0,0,0],[[1,2,3,4]]]	[[1,1,0,-2],[[1,4,2,3]]]
$\rho\gamma$	[[0,0,0,0],[[1,4,3,2]]]	[[0,2,-1,-1],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,3,2,4]]]
ρ^2	[[1,-1,0,0],[[1,2,3,4]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[1,-1,0,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[1,0,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,3,4 ,2]]]	[[1,0,0,-1],[[1,4,3,2]]]	[[0,1,0,-1],[[1,2,4,3]]]
ρ^3	[[1,0,-1,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[1,0,-1,0],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,2,4]]]
$\rho^3\gamma$	[[0,1,0,-1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,1,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,4,2,3]]]

8.3. *Case* $l = 4$

For the epimorphism $\xi_{(0,1,0)}$ we have:

	$\xi_{(0,1,0)}(\sigma_1)$	$\xi_{(0,1,0)}(\sigma_2)$	$\xi_{(0,1,0)}(\sigma_3)$	$\xi_{(0,1,0)}(\sigma_4)$
id	[[0,-1,1,0],[[1,2,3,4]]]	[[-1,0,1,0],[[1,3,4,2]]]	[[0,-1,1,0],[[1,2,3,4]]]	[[-1,0,1,0],[[1,3,4,2]]]
γ	[[1,1,0,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,2,4 ,3]]]	[[1,1,0,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,2,4,3]]]
ρ	[[1,-1,-1,1],[[1,2,3,4]]]	[[1,-1,-1,1],[[1,3 ,2,4]]]	[[1,-1,-1,1],[[1,2,3,4]]]	[[1,-1,-1,1],[[1,3,2,4]]]
$\rho\gamma$	[[-1,1,1,-1],[[1,4,3,2]]]	[[-1,1,1,-1],[[1,4 ,2,3]]]	[[-1,1,1,-1],[[1,4,3,2]]]	[[-1,1,1,-1],[[1,4,2,3]]]
ρ^2	[[2,0,-1,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,2 ,4,3]]]	[[2,0,-1,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,-1,1,0],[[1,4,3,2]]]	[[0,-1,0,1],[[1,3,4 ,2]]]	[[0,-1,1,0],[[1,4,3,2]]]	[[0,-1,0,1],[[1,3,4,2]]]
ρ^3	[[0,1,0,-1],[[1,2,3,4]]]	[[0,2,0,-2],[[1,4,2 ,3]]]	[[0,1,0,-1],[[1,2,3,4]]]	[[0,2,0,-2],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,0,-1,0],[[1,4,3,2]]]	[[2,0,-2,0],[[1,3,2 ,4]]]	[[1,0,-1,0],[[1,4,3,2]]]	[[2,0,-2,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(0,-1,0)}$ we have:

	$\xi_{(0,-1,0)}(\sigma_1)$	$\xi_{(0,-1,0)}(\sigma_2)$	$\xi_{(0,-1,0)}(\sigma_3)$	$\xi_{(0,-1,0)}(\sigma_4)$
id	[[0,1,-1,0],[[1,2,3,4]]]	[[1,0,-1,0],[[1,3,4,2]]]	[[0,1,-1,0],[[1,2,3,4]]]	[[1,0,-1,0],[[1,3,4,2]]]
γ	[[-1,1,0,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[-1,1,0,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,2,4,3]]]
ρ	[[1,-1,1,-1],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,3 ,2,4]]]	[[1,-1,1,-1],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,3,2,4]]]
$\rho\gamma$	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,4 ,2,3]]]	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,4,2,3]]]
ρ^2	[[0,0,-1,1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,2,4 ,3]]]	[[0,0,-1,1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,1,-1,0],[[1,4,3,2]]]	[[0,1,0,-1],[[1,3,4 ,2]]]	[[0,1,-1,0],[[1,4,3,2]]]	[[0,1,0,-1],[[1,3,4,2]]]
ρ^3	[[2,-1,0,-1],[[1,2,3,4]]]	[[0,0,0,0],[[1,4,2 ,3]]]	[[2,-1,0,-1],[[1,2,3,4]]]	[[0,0,0,0],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,0,1,-2],[[1,4,3,2]]]	[[0,0,0,0],[[1,3,2, 4]]]	[[1,0,1,-2],[[1,4,3,2]]]	[[0,0,0,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(0,1,-1)}$ we have:

	$\xi_{(0,1,-1)}(\sigma_1)$	$\xi_{(0,1,-1)}(\sigma_2)$	$\xi_{(0,1,-1)}(\sigma_3)$	$\xi_{(0,1,-1)}(\sigma_4)$
id	[[0,0,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,3,4,2]]]	[[0,0,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,3,4,2]]]
γ	[[0,1,1,-2],[[1,4,3,2]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[0,1,1,-2],[[1,4,3,2]]]	[[1,0,1,-2],[[1,2,4,3]]]
ρ	[[0,-1,0,1],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,-1,0,1],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,2,4]]]
$\rho\gamma$	[[-1,0,1,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[-1,0,1,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,4,2,3]]]
ρ^2	[[2,-1,-1,0],[[1,2,3,4]]]	[[2,-1,0,-1],[[1,2 ,4,3]]]	[[2,-1,-1,0],[[1,2,3,4]]]	[[2,-1,0,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[1,-1,0,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3,4 ,2]]]	[[1,-1,0,0],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3,4,2]]]
ρ^3	[[1,1,-1,-1],[[1,2,3,4]]]	[[0,2,-1,-1],[[1,4 ,2,3]]]	[[1,1,-1,-1],[[1,2,3,4]]]	[[0,2,-1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,1,-1,-1],[[1,4,3,2]]]	[[1,1,-2,0],[[1,3, 2,4]]]	[[1,1,-1,-1],[[1,4,3,2]]]	[[1,1,-2,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(0,-1,1)}$ we have:

	$\xi_{(0,-1,1)}(\sigma_1)$	$\xi_{(0,-1,1)}(\sigma_2)$	$\xi_{(0,-1,1)}(\sigma_3)$	$\xi_{(0,-1,1)}(\sigma_4)$
id	[[0,0,-1,1],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,4,2]]]	[[0,0,-1,1],[[1,2,3,4]]]	[[0,0,-1,1],[[1,3,4,2]]]
γ	[[0,1,-1,0],[[1,4,3,2]]]	[[1,0,-1,0],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,4,3,2]]]	[[1,0,-1,0],[[1,2,4,3]]]
ρ	[[2,-1,0,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,0,-1],[[1,2,3,4]]]	[[2,0,-1,-1],[[1,3,2,4]]]
$\rho\gamma$	[[1,0,1,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[1,0,1,-2],[[1,4,3,2]]]	[[1,1,0,-2],[[1,4,2,3]]]
ρ^2	[[0,1,-1,0],[[1,2,3,4]]]	[[0,1,0,-1],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,2,3,4]]]	[[0,1,0,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[-1,1,0,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,3,4 ,2]]]	[[-1,1,0,0],[[1,4,3,2]]]	[[-1,1,0,0],[[1,3,4,2]]]
ρ^3	[[1,-1,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,4, 2,3]]]	[[1,-1,1,-1],[[1,2,3,4]]]	[[0,0,1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3, 2,4]]]	[[1,-1,1,-1],[[1,4,3,2]]]	[[1,-1,0,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(1,0,0)}$ we have:

	$\xi_{(1,0,0)}(\sigma_1)$	$\xi_{(1,0,0)}(\sigma_2)$	$\xi_{(1,0,0)}(\sigma_3)$	$\xi_{(1,0,0)}(\sigma_4)$
id	[[-1,0,0,1],[[1,2,3,4]]]	[[-1,1,0,0],[[1,3,4,2]]]	[[-1,0,0,1],[[1,2,3,4]]]	[[-1,1,0,0],[[1,3,4,2]]]
γ	[[0,2,-1,-1],[[1,4,3,2]]]	[[0,1,0,-1],[[1,2, 4,3]]]	[[0,2,-1,-1],[[1,4,3,2]]]	[[0,1,0,-1],[[1,2,4,3]]]
ρ	[[2,-2,0,0],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,2 ,4]]]	[[2,-2,0,0],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,2,4]]]
$\rho\gamma$	[[0,0,2,-2],[[1,4,3,2]]]	[[0,0,1,-1],[[1,4,2 ,3]]]	[[0,0,2,-2],[[1,4,3,2]]]	[[0,0,1,-1],[[1,4,2,3]]]
ρ^2	[[1,1,-2,0],[[1,2,3,4]]]	[[1,0,-1,0],[[1,2,4 ,3]]]	[[1,1,-2,0],[[1,2,3,4]]]	[[1,0,-1,0],[[1,2,4,3]]]
$\rho^2\gamma$	[[-1,0,0,1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,4 ,2]]]	[[-1,0,0,1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,4,2]]]
ρ^3	[[1,0,1,-2],[[1,2,3,4]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[1,0,1,-2],[[1,2,3,4]]]	[[1,1,0,-2],[[1,4,2,3]]]
$\rho^3\gamma$	[[2,-1,0,-1],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,0,-1],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,3,2,4]]]

For the epimorphism $\xi_{(-1,0,0)}$ we have:

	$\xi_{(-1,0,0)}(\sigma_1)$	$\xi_{(-1,0,0)}(\sigma_2)$	$\xi_{(-1,0,0)}(\sigma_3)$	$\xi_{(-1,0,0)}(\sigma_4)$
id	[[1,0,0,-1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,4,2]]]	[[1,0,0,-1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,3,4,2]]]
γ	[[0,0,1,-1],[[1,4,3,2]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[0,0,1,-1],[[1,4,3,2]]]	[[2,-1,0,-1],[[1,2,4,3]]]
ρ	[[0,0,0,0],[[1,2,3,4]]]	[[1,1,-2,0],[[1,3,2, 4]]]	[[0,0,0,0],[[1,2,3,4]]]	[[1,1,-2,0],[[1,3,2,4]]]
$\rho\gamma$	[[0,0,0,0],[[1,4,3,2]]]	[[0,2,-1,-1],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,4,3,2]]]	[[0,2,-1,-1],[[1,4,2,3]]]
ρ^2	[[1,-1,0,0],[[1,2,3,4]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[1,-1,0,0],[[1,2,3,4]]]	[[1,0,1,-2],[[1,2,4,3]]]
$\rho^2\gamma$	[[1,0,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,3,4 ,2]]]	[[1,0,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,3,4,2]]]
ρ^3	[[1,0,-1,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[1,0,-1,0],[[1,2,3,4]]]	[[-1,1,0,0],[[1,4,2,3]]]
$\rho^3\gamma$	[[0,1,0,-1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,1,0,-1],[[1,4,3,2]]]	[[0,0,-1,1],[[1,3,2,4]]]

For the epimorphism $\xi_{(-1,0,1)}$ we have:

	$\xi_{(-1,0,1)}(\sigma_1)$	$\xi_{(-1,0,1)}(\sigma_2)$	$\xi_{(-1,0,1)}(\sigma_3)$	$\xi_{(-1,0,1)}(\sigma_4)$
id	[[1,-1,0,0],[[1,2,3,4]]]	[[0,-1,0,1],[[1,3,4,2]]]	[[1,-1,0,0],[[1,2,3,4]]]	[[0,-1,0,1],[[1,3,4,2]]]
γ	[[1,0,0,-1],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,2, 4,3]]]	[[1,0,0,-1],[[1,4,3,2]]]	[[2,0,-1,-1],[[1,2,4,3]]]
ρ	[[1,0,-1,0],[[1,2,3,4]]]	[[2,0,-2,0],[[1,3,2 ,4]]]	[[1,0,-1,0],[[1,2,3,4]]]	[[2,0,-2,0],[[1,3,2,4]]]
$\rho\gamma$	[[0,1,0,-1],[[1,4,3,2]]]	[[0,2,0,-2],[[1,4,2 ,3]]]	[[0,1,0,-1],[[1,4,3,2]]]	[[0,2,0,-2],[[1,4,2,3]]]
ρ^2	[[1,0,0,-1],[[1,2,3,4]]]	[[1,1,0,-2],[[1,2,4 ,3]]]	[[1,0,0,-1],[[1,2,3,4]]]	[[1,1,0,-2],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,0,1,-1],[[1,4,3,2]]]	[[-1,0,1,0],[[1,3,4 ,2]]]	[[0,0,1,-1],[[1,4,3,2]]]	[[-1,0,1,0],[[1,3,4,2]]]
ρ^3	[[0,0,0,0],[[1,2,3,4]]]	[[-1,1,1,-1],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,2,3,4]]]	[[-1,1,1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[0,0,0,0],[[1,4,3,2]]]	[[1,-1,-1,1],[[1,3,2 ,4]]]	[[0,0,0,0],[[1,4,3,2]]]	[[1,-1,-1,1],[[1,3,2,4]]]

For the epimorphism $\xi_{(1,0,-1)}$ we have:

	$\xi_{(1,0,-1)}(\sigma_1)$	$\xi_{(1,0,-1)}(\sigma_2)$	$\xi_{(1,0,-1)}(\sigma_3)$	$\xi_{(1,0,-1)}(\sigma_4)$
id	[[-1,1,0,0],[[1,2,3,4]]]	[[0,1,0,-1],[[1,3,4,2]]]	[[-1,1,0,0],[[1,2,3,4]]]	[[0,1,0,-1],[[1,3,4,2]]]
γ	[[-1,2,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,2, 4,3]]]	[[-1,2,0,-1],[[1,4,3,2]]]	[[0,0,1,-1],[[1,2,4,3]]]
ρ	[[1,-2,1,0],[[1,2,3,4]]]	[[0,0,0,0],[[1,3,2, 4]]]	[[1,-2,1,0],[[1,2,3,4]]]	[[0,0,0,0],[[1,3,2,4]]]
$\rho\gamma$	[[0,-1,2,-1],[[1,4,3,2]]]	[[0,0,0,0],[[1,4,2 ,3]]]	[[0,-1,2,-1],[[1,4,3,2]]]	[[0,0,0,0],[[1,4,2,3]]]
ρ^2	[[1,0,-2,1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[1,0,-2,1],[[1,2,3,4]]]	[[1,-1,0,0],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,0,-1,1],[[1,4,3,2]]]	[[1,0,-1,0],[[1,3,4 ,2]]]	[[0,0,-1,1],[[1,4,3,2]]]	[[1,0,-1,0],[[1,3,4,2]]]
ρ^3	[[2,0,0,-2],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,4, 2,3]]]	[[2,0,0,-2],[[1,2,3,4]]]	[[1,1,-1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[2,0,0,-2],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,3, 2,4]]]	[[2,0,0,-2],[[1,4,3,2]]]	[[1,1,-1,-1],[[1,3,2,4]]]

8.4. *Case* $l = 5$

For the epimorphism $\xi_{(x_1,x_2,x_3,x_4)}$ we have:

	$\xi_{(x_1,x_2,x_3,x_4)}(\sigma_1)$	$\xi_{(x_1,x_2,x_3,x_4)}(\sigma_2)$
id	[[$-x_1 - 2x_3, x_1, x_3, x_3$], [[1, 2]]]	[[$x_3, x_4, -2x_3 - x_4, x_3$], [[2, 3]]]
γ	[[$-x_1, x_1 + 2x_3, -x_3, -x_3$], [[1, 2]]]	[[$-x_4 + 1, -x_3, -x_3, 2x_3 + x_4 - 1$], [[1, 4]]]
ρ	[[$x_3, -x_1 - 2x_3, x_1, x_3$], [[2, 3]]]	[[$x_3, x_3, x_4, -2x_3 - x_4$], [[3, 4]]]
$\rho\gamma$	[[$x_1 + 2x_3 + 1, -x_3, -x_3, -x_1 - 1$], [[1, 4]]]	[[$-x_3, -x_3, 2x_3 + x_4, -x_4$], [[3, 4]]]
ρ^2	[[$x_3, x_3, -x_1 - 2x_3, x_1$], [[3, 4]]]	[[$-2x_3 - x_4 + 1, x_3, x_3, x_4 - 1$], [[1, 4]]]
$\rho^2\gamma$	[[$-x_3, -x_3, -x_1, x_1 + 2x_3$], [[3, 4]]]	[[$-x_3, 2x_3 + x_4, -x_4, -x_3$], [[2, 3]]]
ρ^3	[[$x_1 + 1, x_3, x_3, -x_1 - 2x_3 - 1$], [[1, 4]]]	[[$x_4, -2x_3 - x_4, x_3, x_3$], [[1, 2]]]
$\rho^3\gamma$	[[$-x_3, -x_1, x_1 + 2x_3, -x_3$], [[2, 3]]]	[[$2x_3 + x_4, -x_4, -x_3, -x_3$], [[1, 2]]]

	$\xi_{(x_1,x_2,x_3,x_4)}(\sigma_2)$	$\xi_{(x_1,x_2,x_3,x_4)}(\sigma_3)$
id	[[$x_3, x_3, -2x_3 - x_2, x_2$], [[3, 4]]]	[[$x_3, x_4, -2x_3 - x_4, x_3$], [[2, 3]]]
γ	[[$-x_3, -x_3, -x_2, 2x_3 + x_2$], [[3, 4]]]	[[$-x_4 + 1, -x_3, -x_3, 2x_3 + x_4 - 1$], [[1, 4]]]
ρ	[[$x_2 + 1, x_3, x_3, -2x_3 - x_2 - 1$], [[1, 4]]]	[[$x_3, x_3, x_4, -2x_3 - x_4$], [[3, 4]]]
$\rho\gamma$	[[$-x_3, -x_2, 2x_3 + x_2, -x_3$], [[2, 3]]]	[[$-x_3, -x_3, 2x_3 + x_4, -x_4$], [[3, 4]]]
ρ^2	[[$-2x_3 - x_2, x_2, x_3, x_3$], [[1, 2]]]	[[$-2x_3 - x_4 + 1, x_3, x_3, x_4 - 1$], [[1, 4]]]
$\rho^2\gamma$	[[$-x_2, 2x_3 + x_2, -x_3, -x_3$], [[1, 2]]]	[[$-x_3, 2x_3 + x_4, -x_4, -x_3$], [[2, 3]]]
ρ^3	[[$x_3, -2x_3 - x_2, x_2, x_3$], [[2, 3]]]	[[$x_4, -2x_3 - x_4, x_3, x_3$], [[1, 2]]]
$\rho^3\gamma$	[[$2x_3 + x_2 + 1, -x_3, -x_3, -x_2 - 1$], [[1, 4]]]	[[$2x_3 + x_4, -x_4, -x_3, -x_3$], [[1, 2]]]

8.5. *Case* $l = 6$

For the epimorphism $\xi_{(0,1,0)}$ we have:

	$\xi_{(0,1,0)}(\sigma_1)$	$\xi_{(0,1,0)}(\sigma_2)$	$\xi_{(0,1,0)}(\sigma_3)$	$\xi_{(0,1,0)}(\sigma_4)$
id	[[1,0,0,-1],[[1,3,2,4]]]	[[1,-1,0,0],[[1,3,4,2]]]	[[1,0,-1,0],[[1,4,2,3]]]	[[1,-1,0,0],[[1,3,4,2]]]
γ	[[1,0,0,-1],[[1,3,2,4]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[1,0,-1,0],[[1,4,2,3]]]	[[2,-1,0,-1],[[1,2,4,3]]]
ρ	[[0,0,0,0],[[1,2,4,3]]]	[[1,1,-2,0],[[1,3,2, 4]]]	[[1,1,-1,-1],[[1,3,4,2]]]	[[1,1,-2,0],[[1,3,2,4]]]
$\rho\gamma$	[[0,0,0,0],[[1,2,4,3]]]	[[0,2,-1,-1],[[1,4,2 ,3]]]	[[1,1,-1,-1],[[1,3,4,2]]]	[[0,2,-1,-1],[[1,4,2,3]]]
ρ^2	[[1,0,0,-1],[[1,4,2,3]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[0,1,0,-1],[[1,3,2,4]]]	[[1,0,1,-2],[[1,2,4,3]]]
$\rho^2\gamma$	[[1,0,0,-1],[[1,4,2,3]]]	[[0,0,1,-1],[[1,3,4 ,2]]]	[[0,1,0,-1],[[1,3,2,4]]]	[[0,0,1,-1],[[1,3,4,2]]]
ρ^3	[[0,1,-1,0],[[1,3,4,2]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[0,-1,1,0],[[1,2,4,3]]]	[[-1,1,0,0],[[1,4,2,3]]]
$\rho^3\gamma$	[[0,1,-1,0],[[1,3,4,2]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,-1,1,0],[[1,2,4,3]]]	[[0,0,-1,1],[[1,3,2,4]]]

For the epimorphism $\xi_{(0,-1,0)}$ we have:

	$\xi_{(0,-1,0)}(\sigma_1)$	$\xi_{(0,-1,0)}(\sigma_2)$	$\xi_{(0,-1,0)}(\sigma_3)$	$\xi_{(0,-1,0)}(\sigma_4)$
id	[[-1,0,0,1],[[1,3,2,4]]]	[[-1,1,0,0],[[1,3,4,2]]]	[[-1,0,1,0],[[1,4,2,3]]]	[[-1,1,0,0],[[1,3,4,2]]]
γ	[[1,2,-2,-1],[[1,3,2,4]]]	[[0,1,0,-1],[[1,2, 4,3]]]	[[1,2,-1,-2],[[1,4,2,3]]]	[[0,1,0,-1],[[1,2,4,3]]]
ρ	[[2,-2,0,0],[[1,2,4,3]]]	[[1,-1,0,0],[[1,3,2 ,4]]]	[[1,-1,-1,1],[[1,3,4,2]]]	[[1,-1,0,0],[[1,3,2,4]]]
$\rho\gamma$	[[0,0,2,-2],[[1,2,4,3]]]	[[0,0,1,-1],[[1,4,2 ,3]]]	[[-1,1,1,-1],[[1,3,4,2]]]	[[0,0,1,-1],[[1,4,2,3]]]
ρ^2	[[1,2,-2,-1],[[1,4,2,3]]]	[[1,0,-1,0],[[1,2, 4,3]]]	[[2,1,-2,-1],[[1,3,2,4]]]	[[1,0,-1,0],[[1,2,4,3]]]
$\rho^2\gamma$	[[-1,0,0,1],[[1,4,2,3]]]	[[0,0,-1,1],[[1,3,4 ,2]]]	[[0,-1,0,1],[[1,3,2,4]]]	[[0,0,-1,1],[[1,3,4,2]]]
ρ^3	[[0,1,1,-2],[[1,3,4,2]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[0,1,1,-2],[[1,2,4,3]]]	[[1,1,0,-2],[[1,4,2,3]]]
$\rho^3\gamma$	[[2,-1,-1,0],[[1,3,4,2]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,-1,0],[[1,2,4,3]]]	[[2,0,-1,-1],[[1,3,2,4]]]

For the epimorphism $\xi_{(1,-1,0)}$ we have:

	$\xi_{(1,-1,0)}(\sigma_1)$	$\xi_{(1,-1,0)}(\sigma_2)$	$\xi_{(1,-1,0)}(\sigma_3)$	$\xi_{(1,-1,0)}(\sigma_4)$
id	[[-1,0,1,0],[[1,3,2,4]]]	[[-1,0,1,0],[[1,3,4,2]]]	[[0,-1,1,0],[[1,4,2,3]]]	[[-1,0,1,0],[[1,3,4,2]]]
γ	[[1,2,-1,-2],[[1,3,2,4]]]	[[1,1,0,-2],[[1,2, 4,3]]]	[[2,1,-1,-2],[[1,4,2,3]]]	[[1,1,0,-2],[[1,2,4,3]]]
ρ	[[1,-2,0,1],[[1,2,4,3]]]	[[1,-1,-1,1],[[1,3, 2,4]]]	[[1,0,-2,1],[[1,3,4,2]]]	[[1,-1,-1,1],[[1,3,2,4]]]
$\rho\gamma$	[[-1,0,2,-1],[[1,2,4,3]]]	[[-1,1,1,-1],[[1,4 ,2,3]]]	[[-1,2,0,-1],[[1,3,4,2]]]	[[-1,1,1,-1],[[1,4,2,3]]]
ρ^2	[[2,1,-2,-1],[[1,4,2,3]]]	[[2,0,-1,-1],[[1,2 ,4,3]]]	[[2,1,-1,-2],[[1,3,2,4]]]	[[2,0,-1,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,-1,0,1],[[1,4,2,3]]]	[[0,-1,0,1],[[1,3,4 ,2]]]	[[0,-1,1,0],[[1,3,2,4]]]	[[0,-1,0,1],[[1,3,4,2]]]
ρ^3	[[0,2,0,-2],[[1,3,4,2]]]	[[0,2,0,-2],[[1,4,2 ,3]]]	[[-1,1,1,-1],[[1,2,4,3]]]	[[0,2,0,-2],[[1,4,2,3]]]
$\rho^3\gamma$	[[2,0,-2,0],[[1,3,4,2]]]	[[2,0,-2,0],[[1,3,2 ,4]]]	[[1,-1,-1,1],[[1,2,4,3]]]	[[2,0,-2,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(-1,1,0)}$ we have:

	$\xi_{(-1,1,0)}(\sigma_1)$	$\xi_{(-1,1,0)}(\sigma_2)$	$\xi_{(-1,1,0)}(\sigma_3)$	$\xi_{(-1,1,0)}(\sigma_4)$
id	[[1,0,-1,0],[[1,3,2,4]]]	[[1,0,-1,0],[[1,3,4,2]]]	[[0,1,-1,0],[[1,4,2,3]]]	[[1,0,-1,0],[[1,3,4,2]]]
γ	[[1,0,-1,0],[[1,3,2,4]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,4,2,3]]]	[[1,-1,0,0],[[1,2,4,3]]]
ρ	[[1,0,0,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,3, 2,4]]]	[[1,0,0,-1],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,3,2,4]]]
$\rho\gamma$	[[1,0,0,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,4, 2,3]]]	[[1,0,0,-1],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,4,2,3]]]
ρ^2	[[0,1,0,-1],[[1,4,2,3]]]	[[0,0,1,-1],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,3,2,4]]]	[[0,0,1,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,1,0,-1],[[1,4,2,3]]]	[[0,1,0,-1],[[1,3,4 ,2]]]	[[0,1,-1,0],[[1,3,2,4]]]	[[0,1,0,-1],[[1,3,4,2]]]
ρ^3	[[0,0,0,0],[[1,3,4,2]]]	[[0,0,0,0],[[1,4,2,3]]]	[[1,-1,1,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,4,2,3]]]
$\rho^3\gamma$	[[0,0,0,0],[[1,3,4,2]]]	[[0,0,0,0],[[1,3,2,4]]]	[[1,-1,1,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(0,0,1)}$ we have:

	$\xi_{(0,0,1)}(\sigma_1)$	$\xi_{(0,0,1)}(\sigma_2)$	$\xi_{(0,0,1)}(\sigma_3)$	$\xi_{(0,0,1)}(\sigma_4)$
id	[[0,1,0,-1],[[1,3,2,4]]]	[[0,0,1,-1],[[1,3,4,2]]]	[[1,0,0,-1],[[1,4,2,3]]]	[[0,0,1,-1],[[1,3,4,2]]]
γ	[[0,1,0,-1],[[1,3,2,4]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[1,0,0,-1],[[1,4,2,3]]]	[[1,0,1,-2],[[1,2,4,3]]]
ρ	[[0,-1,1,0],[[1,2,4,3]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,1,-1,0],[[1,3,4,2]]]	[[0,0,-1,1],[[1,3,2,4]]]
$\rho\gamma$	[[0,-1,1,0],[[1,2,4,3]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[0,1,-1,0],[[1,3,4,2]]]	[[-1,1,0,0],[[1,4,2,3]]]
ρ^2	[[1,0,-1,0],[[1,4,2,3]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[1,0,0,-1],[[1,3,2,4]]]	[[2,-1,0,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[1,0,-1,0],[[1,4,2,3]]]	[[1,-1,0,0],[[1,3,4 ,2]]]	[[1,0,0,-1],[[1,3,2,4]]]	[[1,-1,0,0],[[1,3,4,2]]]
ρ^3	[[1,1,-1,-1],[[1,3,4,2]]]	[[0,2,-1,-1],[[1,4 ,2,3]]]	[[0,0,0,0],[[1,2,4,3]]]	[[0,2,-1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,1,-1,-1],[[1,3,4,2]]]	[[1,1,-2,0],[[1,3, 2,4]]]	[[0,0,0,0],[[1,2,4,3]]]	[[1,1,-2,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(0,0,-1)}$ we have:

	$\xi_{(0,0,-1)}(\sigma_1)$	$\xi_{(0,0,-1)}(\sigma_2)$	$\xi_{(0,0,-1)}(\sigma_3)$	$\xi_{(0,0,-1)}(\sigma_4)$
id	[[0,-1,0,1],[[1,3,2,4]]]	[[0,0,-1,1],[[1,3,4,2]]]	[[-1,0,0,1],[[1,4,2,3]]]	[[0,0,-1,1],[[1,3,4,2]]]
γ	[[2,1,-2,-1],[[1,3,2,4]]]	[[1,0,-1,0],[[1,2, 4,3]]]	[[1,2,-2,-1],[[1,4,2,3]]]	[[1,0,-1,0],[[1,2,4,3]]]
ρ	[[2,-1,-1,0],[[1,2,4,3]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,-1,0],[[1,3,4,2]]]	[[2,0,-1,-1],[[1,3,2,4]]]
$\rho\gamma$	[[0,1,1,-2],[[1,2,4,3]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[0,1,1,-2],[[1,3,4,2]]]	[[1,1,0,-2],[[1,4,2,3]]]
ρ^2	[[1,2,-1,-2],[[1,4,2,3]]]	[[0,1,0,-1],[[1,2, 4,3]]]	[[1,2,-2,-1],[[1,3,2,4]]]	[[0,1,0,-1],[[1,2,4,3]]]
$\rho^2\gamma$	[[-1,0,1,0],[[1,4,2,3]]]	[[-1,1,0,0],[[1,3,4 ,2]]]	[[-1,0,0,1],[[1,3,2,4]]]	[[-1,1,0,0],[[1,3,4,2]]]
ρ^3	[[-1,1,1,-1],[[1,3,4,2]]]	[[0,0,1,-1],[[1,4, 2,3]]]	[[0,0,2,-2],[[1,2,4,3]]]	[[0,0,1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,-1,-1,1],[[1,3,4,2]]]	[[1,-1,0,0],[[1,3, 2,4]]]	[[2,-2,0,0],[[1,2,4,3]]]	[[1,-1,0,0],[[1,3,2,4]]]

For the epimorphism $\xi_{(-1,0,1)}$ we have:

	$\xi_{(-1,0,1)}(\sigma_1)$	$\xi_{(-1,0,1)}(\sigma_2)$	$\xi_{(-1,0,1)}(\sigma_3)$	$\xi_{(-1,0,1)}(\sigma_4)$
id	[[0,1,-1,0],[[1,3,2,4]]]	[[0,1,0,-1],[[1,3,4,2]]]	[[0,1,0,-1],[[1,4,2,3]]]	[[0,1,0,-1],[[1,3,4,2]]]
γ	[[0,1,-1,0],[[1,3,2,4]]]	[[0,0,1,-1],[[1,2,4 ,3]]]	[[0,1,0,-1],[[1,4,2,3]]]	[[0,0,1,-1],[[1,2,4,3]]]
ρ	[[1,-1,1,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,3,2 ,4]]]	[[0,0,0,0],[[1,3,4,2]]]	[[0,0,0,0],[[1,3,2,4]]]
$\rho\gamma$	[[1,-1,1,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,3,4,2]]]	[[0,0,0,0],[[1,4,2,3]]]
ρ^2	[[0,1,-1,0],[[1,4,2,3]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[1,0,-1,0],[[1,3,2,4]]]	[[1,-1,0,0],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,1,-1,0],[[1,4,2,3]]]	[[1,0,-1,0],[[1,3,4 ,2]]]	[[1,0,-1,0],[[1,3,2,4]]]	[[1,0,-1,0],[[1,3,4,2]]]
ρ^3	[[1,0,0,-1],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,4, 2,3]]]	[[1,0,0,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,0,0,-1],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,3, 2,4]]]	[[1,0,0,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,3,2,4]]]

For the epimorphism $\xi_{(1,0,-1)}$ we have:

	$\xi_{(1,0,-1)}(\sigma_1)$	$\xi_{(1,0,-1)}(\sigma_2)$	$\xi_{(1,0,-1)}(\sigma_3)$	$\xi_{(1,0,-1)}(\sigma_4)$
id	[[0,-1,1,0],[[1,3,2,4]]]	[[0,-1,0,1],[[1,3,4,2]]]	[[0,-1,0,1],[[1,4,2,3]]]	[[0,-1,0,1],[[1,3,4,2]]]
γ	[[2,1,-1,-2],[[1,3,2,4]]]	[[2,0,-1,-1],[[1,2 ,4,3]]]	[[2,1,-2,-1],[[1,4,2,3]]]	[[2,0,-1,-1],[[1,2,4,3]]]
ρ	[[1,-1,-1,1],[[1,2,4,3]]]	[[2,0,-2,0],[[1,3, 2,4]]]	[[2,0,-2,0],[[1,3,4,2]]]	[[2,0,-2,0],[[1,3,2,4]]]
$\rho\gamma$	[[-1,1,1,-1],[[1,2,4,3]]]	[[0,2,0,-2],[[1,4, 2,3]]]	[[0,2,0,-2],[[1,3,4,2]]]	[[0,2,0,-2],[[1,4,2,3]]]
ρ^2	[[2,1,-1,-2],[[1,4,2,3]]]	[[1,1,0,-2],[[1,2, 4,3]]]	[[1,2,-1,-2],[[1,3,2,4]]]	[[1,1,0,-2],[[1,2,4,3]]]
$\rho^2\gamma$	[[0,-1,1,0],[[1,4,2,3]]]	[[-1,0,1,0],[[1,3,4 ,2]]]	[[-1,0,1,0],[[1,3,2,4]]]	[[-1,0,1,0],[[1,3,4,2]]]
ρ^3	[[-1,2,0,-1],[[1,3,4,2]]]	[[-1,1,1,-1],[[1,4 ,2,3]]]	[[-1,0,2,-1],[[1,2,4,3]]]	[[-1,1,1,-1],[[1,4,2,3]]]
$\rho^3\gamma$	[[1,0,-2,1],[[1,3,4,2]]]	[[1,-1,-1,1],[[1,3, 2,4]]]	[[1,-2,0,1],[[1,2,4,3]]]	[[1,-1,-1,1],[[1,3,2,4]]]

8.6. *Case* $l = 7$

For the epimorphism $\xi_{(0,1,0)}$ we have:

	$\xi_{(0,1,0)}(\sigma_1)$	$\xi_{(0,1,0)}(\sigma_2)$	$\xi_{(0,1,0)}(\sigma_3)$	$\xi_{(0,1,0)}(\sigma_4)$
id	[[1,0,0,-1],[[1,3,2,4]]]	[[1,-1,0,0],[[1,3,4,2]]]	[[1,0,-1,0],[[1,4,2,3]]]	[[1,0,-1,0],[[1,2,4,3]]]
γ	[[1,0,0,-1],[[1,3,2,4]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[1,0,-1,0],[[1,4,2,3]]]	[[0,0,-1,1],[[1,3,4,2]]]
ρ	[[0,0,0,0],[[1,2,4,3]]]	[[1,1,-2,0],[[1,3,2, 4]]]	[[1,1,-1,-1],[[1,3,4,2]]]	[[1,1,0,-2],[[1,4,2,3]]]
$\rho\gamma$	[[0,0,0,0],[[1,2,4,3]]]	[[0,2,-1,-1],[[1,4,2 ,3]]]	[[1,1,-1,-1],[[1,3,4,2]]]	[[2,0,-1,-1],[[1,3,2,4]]]
ρ^2	[[1,0,0,-1],[[1,4,2,3]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[0,1,0,-1],[[1,3,2,4]]]	[[-1,1,0,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[1,0,0,-1],[[1,4,2,3]]]	[[0,0,1,-1],[[1,3,4 ,2]]]	[[0,1,0,-1],[[1,3,2,4]]]	[[0,1,0,-1],[[1,2,4,3]]]
ρ^3	[[0,1,-1,0],[[1,3,4,2]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[0,-1,1,0],[[1,2,4,3]]]	[[1,-1,0,0],[[1,3,2,4]]]
$\rho^3\gamma$	[[0,1,-1,0],[[1,3,4,2]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,-1,1,0],[[1,2,4,3]]]	[[0,0,1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,-1,0)}$ we have:

	$\xi_{(0,-1,0)}(\sigma_1)$	$\xi_{(0,-1,0)}(\sigma_2)$	$\xi_{(0,-1,0)}(\sigma_3)$	$\xi_{(0,-1,0)}(\sigma_4)$
id	[[-1,0,0,1],[[1,3,2,4]]]	[[-1,1,0,0],[[1,3,4,2]]]	[[-1,0,1,0],[[1,4,2,3]]]	[[-1,0,1,0],[[1,2,4,3]]]
γ	[[1,2,-2,-1],[[1,3,2,4]]]	[[0,1,0,-1],[[1,2, 4,3]]]	[[1,2,-1,-2],[[1,4,2,3]]]	[[0,2,-1,-1],[[1,3,4,2]]]
ρ	[[2,-2,0,0],[[1,2,4,3]]]	[[1,-1,0,0],[[1,3,2 ,4]]]	[[1,-1,-1,1],[[1,3,4,2]]]	[[1,-1,0,0],[[1,4,2,3]]]
$\rho\gamma$	[[0,0,2,-2],[[1,2,4,3]]]	[[0,0,1,-1],[[1,4,2 ,3]]]	[[-1,1,1,-1],[[1,3,4,2]]]	[[0,0,1,-1],[[1,3,2,4]]]
ρ^2	[[1,2,-2,-1],[[1,4,2,3]]]	[[1,0,-1,0],[[1,2, 4,3]]]	[[2,1,-2,-1],[[1,3,2,4]]]	[[1,1,-2,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[-1,0,0,1],[[1,4,2,3]]]	[[0,0,-1,1],[[1,3,4 ,2]]]	[[0,-1,0,1],[[1,3,2,4]]]	[[0,-1,0,1],[[1,2,4,3]]]
ρ^3	[[0,1,1,-2],[[1,3,4,2]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[0,1,1,-2],[[1,2,4,3]]]	[[1,1,0,-2],[[1,3,2,4]]]
$\rho^3\gamma$	[[2,-1,-1,0],[[1,3,4,2]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,-1,0],[[1,2,4,3]]]	[[2,0,-1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(1,-1,0)}$ we have:

	$\xi_{(1,-1,0)}(\sigma_1)$	$\xi_{(1,-1,0)}(\sigma_2)$	$\xi_{(1,-1,0)}(\sigma_3)$	$\xi_{(1,-1,0)}(\sigma_4)$
id	[[0,-1,0,1],[[1,3,2,4]]]	[[0,0,-1,1],[[1,3,4,2]]]	[[-1,0,0,1],[[1,4,2,3]]]	[[0,-1,0,1],[[1,2,4,3]]]
γ	[[2,1,-2,-1],[[1,3,2,4]]]	[[1,0,-1,0],[[1,2, 4,3]]]	[[1,2,-2,-1],[[1,4,2,3]]]	[[1,1,-2,0],[[1,3,4,2]]]
ρ	[[2,-1,-1,0],[[1,2,4,3]]]	[[2,0,-1,-1],[[1,3 ,2,4]]]	[[2,-1,-1,0],[[1,3,4,2]]]	[[2,0,-1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[0,1,1,-2],[[1,2,4,3]]]	[[1,1,0,-2],[[1,4,2 ,3]]]	[[0,1,1,-2],[[1,3,4,2]]]	[[1,1,0,-2],[[1,3,2,4]]]
ρ^2	[[1,2,-1,-2],[[1,4,2,3]]]	[[0,1,0,-1],[[1,2, 4,3]]]	[[1,2,-2,-1],[[1,3,2,4]]]	[[0,2,-1,-1],[[1,3,4,2]]]
$\rho^2\gamma$	[[-1,0,1,0],[[1,4,2,3]]]	[[-1,1,0,0],[[1,3,4 ,2]]]	[[-1,0,0,1],[[1,3,2,4]]]	[[-1,0,1,0],[[1,2,4,3]]]
ρ^3	[[-1,1,1,-1],[[1,3,4,2]]]	[[0,0,1,-1],[[1,4, 2,3]]]	[[0,0,2,-2],[[1,2,4,3]]]	[[0,0,1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,-1,-1,1],[[1,3,4,2]]]	[[1,-1,0,0],[[1,3, 2,4]]]	[[2,-2,0,0],[[1,2,4,3]]]	[[1,-1,0,0],[[1,4,2,3]]]

For the epimorphism $\xi_{(-1,1,0)}$ we have:

	$\xi_{(-1,1,0)}(\sigma_1)$	$\xi_{(-1,1,0)}(\sigma_2)$	$\xi_{(-1,1,0)}(\sigma_3)$	$\xi_{(-1,1,0)}(\sigma_4)$
id	[[0,1,0,-1],[[1,3,2,4]]]	[[0,0,1,-1],[[1,3,4,2]]]	[[1,0,0,-1],[[1,4,2,3]]]	[[0,1,0,-1],[[1,2,4,3]]]
γ	[[0,1,0,-1],[[1,3,2,4]]]	[[1,0,1,-2],[[1,2,4 ,3]]]	[[1,0,0,-1],[[1,4,2,3]]]	[[-1,1,0,0],[[1,3,4,2]]]
ρ	[[0,-1,1,0],[[1,2,4,3]]]	[[0,0,-1,1],[[1,3,2 ,4]]]	[[0,1,-1,0],[[1,3,4,2]]]	[[0,0,1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[0,-1,1,0],[[1,2,4,3]]]	[[-1,1,0,0],[[1,4,2 ,3]]]	[[0,1,-1,0],[[1,3,4,2]]]	[[1,-1,0,0],[[1,3,2,4]]]
ρ^2	[[1,0,-1,0],[[1,4,2,3]]]	[[2,-1,0,-1],[[1,2, 4,3]]]	[[1,0,0,-1],[[1,3,2,4]]]	[[0,0,-1,1],[[1,3,4,2]]]
$\rho^2\gamma$	[[1,0,-1,0],[[1,4,2,3]]]	[[1,-1,0,0],[[1,3,4 ,2]]]	[[1,0,0,-1],[[1,3,2,4]]]	[[1,0,-1,0],[[1,2,4,3]]]
ρ^3	[[1,1,-1,-1],[[1,3,4,2]]]	[[0,2,-1,-1],[[1,4 ,2,3]]]	[[0,0,0,0],[[1,2,4,3]]]	[[2,0,-1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,1,-1,-1],[[1,3,4,2]]]	[[1,1,-2,0],[[1,3, 2,4]]]	[[0,0,0,0],[[1,2,4,3]]]	[[1,1,0,-2],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,0,1)}$ we have:

	$\xi_{(0,0,1)}(\sigma_1)$	$\xi_{(0,0,1)}(\sigma_2)$	$\xi_{(0,0,1)}(\sigma_3)$	$\xi_{(0,0,1)}(\sigma_4)$
id	[[1,0,-1,0],[[1,3,2,4]]]	[[1,0,-1,0],[[1,3,4,2]]]	[[0,1,-1,0],[[1,4,2,3]]]	[[0,0,-1,1],[[1,2,4,3]]]
γ	[[1,0,-1,0],[[1,3,2,4]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,4,2,3]]]	[[0,1,-2,1],[[1,3,4,2]]]
ρ	[[1,0,0,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,3, 2,4]]]	[[1,0,0,-1],[[1,3,4,2]]]	[[2,0,0,-2],[[1,4,2,3]]]
$\rho\gamma$	[[1,0,0,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,4, 2,3]]]	[[1,0,0,-1],[[1,3,4,2]]]	[[2,0,0,-2],[[1,3,2,4]]]
ρ^2	[[0,1,0,-1],[[1,4,2,3]]]	[[0,0,1,-1],[[1,2,4 ,3]]]	[[0,1,-1,0],[[1,3,2,4]]]	[[-1,2,-1,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,1,0,-1],[[1,4,2,3]]]	[[0,1,0,-1],[[1,3,4 ,2]]]	[[0,1,-1,0],[[1,3,2,4]]]	[[-1,1,0,0],[[1,2,4,3]]]
ρ^3	[[0,0,0,0],[[1,3,4,2]]]	[[0,0,0,0],[[1,4,2,3]]]	[[1,-1,1,-1],[[1,2,4,3]]]	[[1,-1,1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[0,0,0,0],[[1,3,4,2]]]	[[0,0,0,0],[[1,3,2,4]]]	[[1,-1,1,-1],[[1,2,4,3]]]	[[1,-1,1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(0,0,-1)}$ we have:

	$\xi_{(0,0,-1)}(\sigma_1)$	$\xi_{(0,0,-1)}(\sigma_2)$	$\xi_{(0,0,-1)}(\sigma_3)$	$\xi_{(0,0,-1)}(\sigma_4)$
id	[[-1,0,1,0],[[1,3,2,4]]]	[[-1,0,1,0],[[1,3,4,2]]]	[[0,-1,1,0],[[1,4,2,3]]]	[[0,0,1,-1],[[1,2,4,3]]]
γ	[[1,2,-1,-2],[[1,3,2,4]]]	[[1,1,0,-2],[[1,2, 4,3]]]	[[2,1,-1,-2],[[1,4,2,3]]]	[[0,1,0,-1],[[1,3,4,2]]]
ρ	[[1,-2,0,1],[[1,2,4,3]]]	[[1,-1,-1,1],[[1,3, 2,4]]]	[[1,0,-2,1],[[1,3,4,2]]]	[[0,0,0,0],[[1,4,2,3]]]
$\rho\gamma$	[[-1,0,2,-1],[[1,2,4,3]]]	[[-1,1,1,-1],[[1,4 ,2,3]]]	[[-1,2,0,-1],[[1,3,4,2]]]	[[0,0,0,0],[[1,3,2,4]]]
ρ^2	[[2,1,-2,-1],[[1,4,2,3]]]	[[2,0,-1,-1],[[1,2 ,4,3]]]	[[2,1,-1,-2],[[1,3,2,4]]]	[[1,0,-1,0],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,-1,0,1],[[1,4,2,3]]]	[[0,-1,0,1],[[1,3,4 ,2]]]	[[0,-1,1,0],[[1,3,2,4]]]	[[1,-1,0,0],[[1,2,4,3]]]
ρ^3	[[0,2,0,-2],[[1,3,4,2]]]	[[0,2,0,-2],[[1,4,2 ,3]]]	[[-1,1,1,-1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,3,2,4]]]
$\rho^3\gamma$	[[2,0,-2,0],[[1,3,4,2]]]	[[2,0,-2,0],[[1,3,2 ,4]]]	[[1,-1,-1,1],[[1,2,4,3]]]	[[1,1,-1,-1],[[1,4,2,3]]]

For the epimorphism $\xi_{(-1,0,1)}$ we have:

	$\xi_{(-1,0,1)}(\sigma_1)$	$\xi_{(-1,0,1)}(\sigma_2)$	$\xi_{(-1,0,1)}(\sigma_3)$	$\xi_{(-1,0,1)}(\sigma_4)$
id	[[0,1,-1,0],[[1,3,2,4]]]	[[0,1,0,-1],[[1,3,4,2]]]	[[0,1,0,-1],[[1,4,2,3]]]	[[-1,1,0,0],[[1,2,4,3]]]
γ	[[0,1,-1,0],[[1,3,2,4]]]	[[0,0,1,-1],[[1,2,4 ,3]]]	[[0,1,0,-1],[[1,4,2,3]]]	[[-1,2,-1,0],[[1,3,4,2]]]
ρ	[[1,-1,1,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,3,2 ,4]]]	[[0,0,0,0],[[1,3,4,2]]]	[[1,-1,1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[1,-1,1,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,4,2 ,3]]]	[[0,0,0,0],[[1,3,4,2]]]	[[1,-1,1,-1],[[1,3,2,4]]]
ρ^2	[[0,1,-1,0],[[1,4,2,3]]]	[[1,-1,0,0],[[1,2,4 ,3]]]	[[1,0,-1,0],[[1,3,2,4]]]	[[0,1,-2,1],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,1,-1,0],[[1,4,2,3]]]	[[1,0,-1,0],[[1,3,4 ,2]]]	[[1,0,-1,0],[[1,3,2,4]]]	[[0,0,-1,1],[[1,2,4,3]]]
ρ^3	[[1,0,0,-1],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,4, 2,3]]]	[[1,0,0,-1],[[1,2,4,3]]]	[[2,0,0,-2],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,0,0,-1],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,3, 2,4]]]	[[1,0,0,-1],[[1,2,4,3]]]	[[2,0,0,-2],[[1,4,2,3]]]

For the epimorphism $\xi_{(1,0,-1)}$ we have:

	$\xi_{(1,0,-1)}(\sigma_1)$	$\xi_{(1,0,-1)}(\sigma_2)$	$\xi_{(1,0,-1)}(\sigma_3)$	$\xi_{(1,0,-1)}(\sigma_4)$
id	[[0,-1,1,0],[[1,3,2,4]]]	[[0,-1,0,1],[[1,3,4,2]]]	[[0,-1,0,1],[[1,4,2,3]]]	[[1,-1,0,0],[[1,2,4,3]]]
γ	[[2,1,-1,-2],[[1,3,2,4]]]	[[2,0,-1,-1],[[1,2 ,4,3]]]	[[2,1,-2,-1],[[1,4,2,3]]]	[[1,0,-1,0],[[1,3,4,2]]]
ρ	[[1,-1,-1,1],[[1,2,4,3]]]	[[2,0,-2,0],[[1,3, 2,4]]]	[[2,0,-2,0],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,4,2,3]]]
$\rho\gamma$	[[-1,1,1,-1],[[1,2,4,3]]]	[[0,2,0,-2],[[1,4, 2,3]]]	[[0,2,0,-2],[[1,3,4,2]]]	[[1,1,-1,-1],[[1,3,2,4]]]
ρ^2	[[2,1,-1,-2],[[1,4,2,3]]]	[[1,1,0,-2],[[1,2, 4,3]]]	[[1,2,-1,-2],[[1,3,2,4]]]	[[0,1,0,-1],[[1,3,4,2]]]
$\rho^2\gamma$	[[0,-1,1,0],[[1,4,2,3]]]	[[-1,0,1,0],[[1,3,4 ,2]]]	[[-1,0,1,0],[[1,3,2,4]]]	[[0,0,1,-1],[[1,2,4,3]]]
ρ^3	[[-1,2,0,-1],[[1,3,4,2]]]	[[-1,1,1,-1],[[1,4 ,2,3]]]	[[-1,0,2,-1],[[1,2,4,3]]]	[[0,0,0,0],[[1,3,2,4]]]
$\rho^3\gamma$	[[1,0,-2,1],[[1,3,4,2]]]	[[1,-1,-1,1],[[1,3, 2,4]]]	[[1,-2,0,1],[[1,2,4,3]]]	[[0,0,0,0],[[1,4,2,3]]]

8.7. Conclusion

In the previous computations, and tables resulting from them, we did not find a repeated line. This means that all epimorphisms presented in this section are different up to automorphisms of $W(\widetilde{A}_n)$ with $n > 1$.

9. The case $A(\widetilde{A}_1)$

In this section we deal with the special case $A(\widetilde{A}_1)$. We state and prove the following result:

Theorem 9.1. *The representatives of the classes of epimorphisms from $A(\widetilde{A}_1)$ to its Coxeter group $W(\widetilde{A}_1)$, are $\xi_1^{w'}$ and ξ_2^w defined by:*

$$\begin{cases} \xi_1^w(\sigma_1) = w \\ \xi_1^w(\sigma_2) = w' \end{cases}$$

where $w = prod_{q'}(s_1 s_2)$ *with* q' *odd,* $|w'| = 2$ *and*

$$\begin{cases} \xi_2^w(\sigma_1) = w' \\ \xi_2^w(\sigma_2) = w \end{cases}$$

where $w = prod_q(s_1 s_2)$ *with* q *odd and* $w' = s_1 s_2$.

Recall that all the automorphisms of $W(\widetilde{A}_1)$ are inner by graph (see [F]). Let ρ denote the automorphism of $W(\widetilde{A}_1)$ the sends s_1 to s_2 and s_2 to s_1.

We will state a few lemmas before.

Lemma 9.2. *Let* ξ*, from* $A(\widetilde{A}_1)$ *to* $W(\widetilde{A}_1)$*, be a morphism such that*

$$\begin{cases} \xi(\sigma_1) = w \\ \xi(\sigma_2) = w' \end{cases}$$

with $|w|, |w'| \geq 3$*. Then* ξ *is not an epimorphism.*

Proof. Suppose that

$$\xi(\sigma_1) = w = s_1 w_1$$

where $w_1 = prod_l(s_2, s_1)$, for some $l \geq 1$. Note that we can still suppose in general that $\xi(\sigma_1)$ starts by s_1 up to ρ. Now suppose that

$$\xi(\sigma_2) = w'$$

where $w' = prod_q(s_2, s_1)$, for some $q > 2$. We will see later that assuming that w' starts by s_2 is not a real restriction. Then

1. If l is even and q is odd: in this case there are no cancelations beside the trivial ones and $|\xi(\sigma_i)\xi(\sigma_j)| = |\xi(\sigma_i)||\xi(\sigma_j) > 1$, for $i, j \in \{1, 2\}$ and ξ is not an epimorphism.
2. If l is even, q is even, $q = l$: Suppose that there ia a word $\omega \in A(\widetilde{A}_1)$ such that $\xi(\omega) = \sigma_2$. Suppose that ω has minimal length, $|\omega|$ (in the generators and their inverses), among all words such that $\xi(\omega) = \sigma_2$ and that $|\omega| \geq 3$. If $|\omega| = 3$ we have the following possibilities for ω:
   - (a) $\omega = \sigma_2^{-1}\sigma_1\sigma_2$, then $|\xi(\omega)| = 3q + 1 > 1$.
   - (b) $\omega = \sigma_2\sigma_1\sigma_2^{-1}$, then $|\xi(\omega)| = q - 1 > 1$.
   - (c) $\omega = \sigma_1\sigma_2\sigma_1$, then $\xi(\omega) = \xi(\sigma_2^{-1})$ and $|\omega|$ is not minimal.
   - (d) $\omega = \sigma_1\sigma_2^{-1}\sigma_1$, then $\xi(\omega) = \xi(\sigma_2)$ and $|\omega|$ is not minimal.

   So ω cannot have length 3. Suppose that $|\omega| = 4$ then
   - (a) $\omega = \sigma_2^{-1}\sigma_1\sigma_2\sigma_2$, then $|\xi(\omega)| = 4q + 1 > q + 2$.
   - (b) $\omega = \sigma_2\sigma_1\sigma_2^{-1}\sigma_2^{-1}$, then $|\xi(\omega)| = 2q - 1 > q + 2$.
   - (c) $\omega = \sigma_2^{-1}\sigma_1\sigma_2\sigma_1$, then $\xi(\omega) = \xi(\sigma_2^{-2})$ and $|\omega|$ is not minimal.

   Now suppose, has our induction hypothesis, that $\omega = \omega_3\omega_1$, $|\omega_3\omega_1| > q + 2$ for $|\omega_3| = 3$ and $|\omega_1| = k > 1$. Let $\omega_2 \in A(\widetilde{A}_1)$ be such that $|\omega_2| = k + 1$. So $\omega_2 = \omega_1 t$ with $|\omega_1| = k$ and $t \in \{\sigma_1, \sigma_2, \sigma_2^{-1}\}$. We have that $|\xi(\omega_1)| > q + 2$ hence for any t the length of $|\xi(\omega_1 t)| > q + 2 - (q + 1) > 1$, being $q + 1$ the biggest amount of cancelations possible.
   We conclude that ξ is not an epimorphism.
3. If l is even, q is even, $q \neq l$ and $q \neq 2$: This is just like the previous case the proof being similar. this is not an epimorphism.
4. If l is even and $q = 2$: We have $\xi(\sigma_1\sigma_2^{-\frac{l}{2}}) = s_1$ and $\xi(\sigma_2^{\frac{l}{2}+1}\sigma_1) = s_2$ and we have an epimorphism.
5. If l is odd and q is even: Then $|\xi(\sigma_i)\xi(\sigma_j)|$ are even, for $i, j \in \{1, 2\}$ and ξ is not an epimorphism.

6. If l is odd, q is odd and $q = l$: This is case 2 in which we change the roles of σ_1 and σ_2.
7. If l is odd, q is odd, $q \neq l$ and $l \neq 1$: This is, again, a previous case 3 in which we change the roles of σ_1 and σ_2.
8. If q is odd and $l = 1$: Finally this is case 4 in which we change the roles of σ_1 and σ_2 and we have an epimorphism.

The case where $w' = prod_q(s_1, s_2)$ is the same. We just have to take in consideration of making the products in the reversed order. We obtain one of the previous cases.

□

Lemma 9.3. *Let ξ, from $A(\widetilde{A}_1)$ to $W(\widetilde{A}_1)$, be a morphism such that*

$$\begin{cases} \xi(\sigma_1) = w \\ \xi(\sigma_2) = w' \end{cases}$$

with $|w| = 1$ and $|w'| \geq 3$. Then ξ is not an epimorphism different from μ up to an automorphism of $W(\widetilde{A}_1)$.

Proof. Let us suppose that $w = s_1$. We will proceed by induction on the length of w'. Suppose now that $|w'| = 3$. If $w' = s_1 s_2 s_1$ then ξ is in the same class of μ up to conjugation by s_1. Let $w' = s_2 s_1 s_2$. The proof is similar to the proof of Lemma 9.2 to show that we cannot have a word of length 2 as image of ξ.

□

Lemma 9.4. *The morphisms from $A(\widetilde{A}_1)$ to its Coxeter group $W(\widetilde{A}_1)$, defined by:*

$$\begin{cases} \xi_1^{w,w'}(\sigma_1) = w \\ \xi_1^{w,w'}(\sigma_2) = w' \end{cases}$$

where $w = prod_{q'}(s_1 s_2)$ with q' odd, $|w'| = 2$ and

$$\begin{cases} \xi_2^{w',w}(\sigma_1) = w' \\ \xi_2^{w',w}(\sigma_2) = w \end{cases}$$

where $w = prod_q(s_1 s_2)$ with q odd and $w' = s_1 s_2$ are epimorphisms. This epimorphisms are different from the standard epimorphism μ.

Proof. Suppose that $w' = s_1 s_2$, then $\xi_1^{w,w'}(\sigma_1(\sigma_2)^{\frac{q'-1}{2}}) = s_1$ and $\xi_1^{w,w'}(\sigma_1(\sigma_2)^{\frac{q'+1}{2}}) = s_2$. This same argument is used in every case showing that this are in fact epimorphisms.

To see that $\xi_1^{w,w'}$ is different from μ notice that supposing that $w' = s_1 s_2$ we have

$$\xi_1^{w,w'}(\sigma_2^2) = s_1 s_2 s_1 s_2$$

and

$$\mu(\sigma_2^2) = id.$$

There is no automorphism of $W(\widetilde{A}_1)$ sending $\xi(\sigma_2^2)$ to id.

□

Consider the following epimorphisms from $A(\widetilde{A}_1)$ to its Coxeter group $W(\widetilde{A}_1)$, defined by:

$$\begin{cases} \xi_1^{w,w'}(\sigma_1) = w \\ \xi_1^{w,w'}(\sigma_2) = w' \end{cases}$$

where $w = prod_{q'}(s_1 s_2)$ with q' odd, $|w'| = 2$ and

$$\begin{cases} \xi_2^{w',w}(\sigma_1) = s_1 s_2 \\ \xi_2^{w',w}(\sigma_2) = w \end{cases}$$

where $w = prod_q(s_1 s_2)$ with q odd.

Lemma 9.5. *Let $\xi_1^{w,w'}$,$\xi_1^{w'',w'}$,$\xi_2^{w',w'''}$ and $\xi_2^{w',w}$ be four epimorphisms defined as above. Then, if $i \neq j$ or $w \neq w''$ or $w' \neq w'''$ then it does not exist an automorphism ψ of $W(\widetilde{A}_1)$ such that $\xi_i^{w,w'} = \psi(\xi_j^{w'',w'''})$*

Proof. Now to see which of this epimorphisms are the same up to an automorphism of $W(\widetilde{A}_1)$. Let ξ_1^w,$\xi_1^{w'}$ be epimorphisms such that $w = prod_q(s_2 s_1) \neq w' = prod_{q'}(s_2 s_1)$ and $q' > q$. Suppose that there is an automorphism of $W(\widetilde{A}_1)$, ψ, such that $\psi(\xi_1^w) = \xi_1^{w'}$. In this case if we compute

$$\psi(\xi_1^w)(\sigma_1) = \psi(s_1 Prod_q(s_2 s_1)) = s_1 Prod_{q'}(s_2 s_1) = \xi_1^{w'}(\sigma_1)$$

and

$$\psi(\xi_1^w)(\sigma_2) = \psi(s_2 s_1) = s_2 s_1 = \xi_1^{w'}(\sigma_2).$$

Hence $\psi(s_1 Prod_q(s_2 s_1)) = \psi(s_1) Prod_q(s_2 s_1)$ if q is even and $\psi(s_1 Prod_q(s_2 s_1)) = \psi(s_1) Prod_{q-1}(s_2 s_1)\psi(s_2)$ if q is odd. Suppose that q is even. In this case we have that $\psi(s_1) = s_1 Prod_{q-q'}(s_2 s_1)$ and $\psi(s_2) = \psi(s_2 s_1)\psi(s_1) = s_1 Prod_{q'-q-2}(s_2 s_1)$. If q' is odd then $|\psi(s_1^2)| = 2|s_1 Prod_{q-q'}(s_2 s_1)|$ and ψ is not an automorphism. So q' must be even. This is, again, not possible because in this case the length of the image of any word by ψ is even.

Suppose now that q is odd. We have

$$\psi(s_1) Prod_{q-1}(s_2 s_1)\psi(s_2) = s_1 Prod_{q'}(s_2 s_1)$$

which is equivalent to

$$\psi(s_2)\psi(s_1) Prod_{q-1}(s_2 s_1)\psi(s_2) = \psi(s_2) s_1 Prod_{q'}(s_2 s_1),$$

$$Prod_{q+1}(s_2 s_1)\psi(s_2) = \psi(s_2) s_1 Prod_{q'}(s_2 s_1)$$

and

$$Prod_{q+1}(s_2 s_1) = \psi(s_2) s_1 Prod_{q'}(s_2 s_1)\psi(s_2).$$

If q' is even this means that $2|\psi(s_2)| = q' - q$ which is false because q' is even, q is odd and $q' - q$ is even. So ψ is not an automorphism.

This means that if $w \neq w'$ then the class of ξ_1^w is different from the class of $\xi_1^{w'}$.

The case of the epimorphisms of type ξ_2^w is analogous the the previous one.

Now we prove that the class of ξ_2^w is different from the class of $\xi_1^{w'}$. Suppose that there is an automorphism ψ such that:

$$\psi(\xi_1^{w'}) = \xi_2^w.$$

We have:

$$\psi(\xi_1^{w'}(\sigma_1)) = \psi(s_1 w') = s_1 s_2 = \xi_2^{w}(\sigma_1)$$

and

$$\psi(\xi_1^{w'}(\sigma_2)) = \psi(s_2 s_1) = w = \xi_2^{w}(\sigma_2).$$

Notice that $w = Prod_q(s_2 s_1)$ with q odd. So $w = w^{-1}$ and $\psi(s_2 s_1 s_2 s_1) = id$ which is impossible (this equality implies that the group $W(\widetilde{A}_1)$ is finite). So there is not such automorphism and we are done.

□

We summarize the proof of the Main Theorem of this section.

Proof of theorem 9.1. Lemmas 9.2 and 9.3 state the shape of the candidates to be epimorphism. Lemma 9.4 shows that the morphisms that are not covered by Lemmas 9.2 and 9.3 are epimorphisms. Finally Lemma 9.5 shows that the epimorphisms of Lemma 9.4 are different up to an automorphism of $W(\widetilde{A}_1)$.

□

10. Main result

So finally we present, in an condensed form, the main result already proved:

Theorem 10.1. *The representatives of the classes of epimorphisms from $A(\widetilde{A}_{n-1})$ to its Coxeter group $W(\widetilde{A}_{n-1})$, for $n > 1$, are:*

1. $\begin{cases} \xi_1^{w,w'}(\sigma_1) = w \\ \xi_1^{w,w'}(\sigma_2) = w' \end{cases}$ *where $w = prod_{q'}(s_1 s_2)$ with q' odd, $|w'| = 2$ and*
2. $\begin{cases} \xi_2^{w',w}(\sigma_1) = s_1 s_2 \\ \xi_2^{w',w}(\sigma_2) = w \end{cases}$ *where $w = prod_q(s_1 s_2)$ with q odd.*
3. $(\xi_1)_{(0,\pm1)}$ *for all $n \geq 3$.*
4. $(\xi_1)_{(y,p)}$ *for $gcd(y,p) = 1$ and $n \geq 3$ odd.*
5. $(\xi_1)_{(y,p)}$ *for $gcd(y,p) = 1$, $n \geq 3$ even and p odd.*
6. $(\xi_3)_{(x_1,x_2,x_3)}$ *for $(x_1,x_2,x_3) \in \{(\pm1,0,0),(\pm1,\mp1,0),(0,0,\pm1),(0,\pm1,\mp1)\}$.*
7. $(\xi_4)_{(x_1,x_2,x_3)}$ *for $(x_1,x_2,x_3) \in \{(0,\pm1,0),(0,\pm1,\mp1),(\pm1,0,0),(\pm1,0,\mp1)\}$.*
8. $(\xi_5)_{(x_1,\ldots,x_4)}$ *for $x_1 + x_3$ odd, $gcd(x_1,x_3,x_4) = gcd(x_4, x_1+x_3+2x_4) = gcd(x_4, x_1 - x_3) = gcd(x_4, x_1+x_2+x_3+2x_4) = 1$.*
9. $(\xi_6)_{(x_1,x_2,x_3)}$ *for $(x_1,x_2,x_3) \in \{(0,\pm1,0),(\pm1,\mp1,0),(0,0,\pm1),(\pm1,0,\mp1)\}$.*
10. $(\xi_7)_{(x_1,x_2,x_3)}$ *for $(x_1,x_2,x_3) \in \{(0,\pm1,0),(\pm1,\mp1,0),(0,0,\pm1),(\pm1,0,\mp1)\}$.*

Where the epimorphisms ξ_k are the ones introduced in the k^{th} subsection of section 3.

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