Efective computation of the multivariable Alexander polinomial of Lorenz links

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Abstract

Given two different representations of a Lorenz link, we compare how they affect the computation of the multivariable Alexander polynomial. We also compare the Alexander polynomial with the trip number and genus. Our experimental results lead us to conjecture that, for Lorenz knots, the Alexander polynomial is an equivalent invariant to the pair (trip number, genus). Finally we give a counterexample in the case of Lorenz links.

Key words: Alexander Polynomial, genus, trip number, Lorenz kots

1 Introduction

We define a *Lorenz flow* as a semi-flow that has a singularity of saddle type with a one-dimensional unstable manifold and an infinite set of hyperbolic periodic orbits, whose closure contains the saddle point (see [7]). A Lorenz flow, together with an extra geometric assumption (see [11]) is called a *Geometric Lorenz flow*. The dynamics of this type of flows can be described by first-return one-dimensional maps with one discontinuity, that are not necessarily surjective in the continuity subintervals. This maps are called Lorenz maps, more precisely, we will adopt the following definition introduced in [7].

Definition 1 *Let $P < 0 < Q$ and $r \geq 1$. A C^r Lorenz map $f : [P, Q] \to [P, Q]$ is a map described by a pair (f_-, f_+) where:*

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(1) $f_- : [P, 0] \to [P, Q]$ *and* $f_+ : [0, Q] \to [P, Q]$*are continuous and strictly increasing maps;*
(2) $f(P) = P$, $f(Q) = Q$ *and* f *has no other fixed points in* $[P, Q]\backslash\{0\}$.
(3) There exists $\rho > 0$, *the exponent of* f, *such that*

$$f_-(x) = \widetilde{f}_-(|x|^\rho) \text{ and } f_+(x) = \widetilde{f}_+(|x|^\rho)$$

where $\widetilde{f}_-$ *and* $\widetilde{f}_+$, *the coefficients of the Lorenz map, are* C^r *diffeomorphisms defined on appropriate closed intervals.*

Because of the ambiguity at the point 0, *we consider the map undefined in* 0. *This Lorenz map is denoted by* (P, Q, f_-, f_+) *(if there is no ambiguity about the interval of definition, we erase the corresponding symbols* P, Q*).*

Let $f^j = f \circ f^{j-1}$, $f^0 = id$, be the j-th iterate of the map f. We define the *itinerary* of a point x under a Lorenz map f as $i_f(x) = (i_f(x))_j, j = 0, 1, \ldots$, where

$$(i_f(x))_j = \begin{cases} L \text{ if } f^j(x) < 0 \\ 0 \text{ if } f^j(x) = 0 \\ R \text{ if } f^j(x) > 0 \end{cases}.$$

It is obvious that the itinerary of a point x will be a finite sequence in the symbols L and R with 0 as its last symbol, if and only if x is a pre-image of 0 and otherwise it is one infinite sequence in the symbols L and R. So it is natural to consider the symbolic space Σ of sequences $X_0 \cdots X_n$ on the symbols $\{L, 0, R\}$, such that $X_i \neq 0$ for all $i < n$ and: $n = \infty$ or $X_n = 0$, with the lexicographic order relation induced by $L < 0 < R$.

It is straightforward to verify that, for all $x, y \in [-1, 1]$, we have

(1) If $x < y$ then $i_f(x) \leq i_f(y)$, and
(2) If $i_f(x) < i_f(y)$ then $x < y$.

We define the *kneading invariant* associated to a Lorenz map $f = (f_-, f_+)$, as

$$K_f = (K_f^-, K_f^+) = (Li_f(f_-(0)), Ri_f(f_+(0))).$$

We say that a pair $(X, Y) \in \Sigma \times \Sigma$ is *admissible* if $(X, Y) = K_f$ for some Lorenz map f.

Consider the *shift map* $s : \Sigma \setminus \{0\} \to \Sigma$, $s(X_0 \cdots X_n) = X_1 \cdots X_n$. The set of admissible pairs is characterized, combinatorially, in the following way (see for example [6]).

Proposition 1 *A pair* $(X,Y)\in\Sigma\times\Sigma$ *is admissible if and only if* $X_0=L$, $Y_0=R$ *and, for* $Z\in\{X,Y\}$ *we have:*
(1) If $Z_i=L$ *then* $s^i(Z)\le X$*;*
(2) If $Z_i=R$ *then* $s^i(Z)\ge Y$*; with inequality (1) (resp. (2)) strict if* X *(resp.* Y*) is finite.*

2 Braids, Lorenz links and the Alexander polynomial

Let $n>0$ be an integer. We denote by B_n the braid group on n strings given by the following presentation (see [1]):

$$B_n=\left\langle \sigma_1,\sigma_2,\ldots,\sigma_{n-1}\;\middle|\;\begin{array}{ll}\sigma_i\sigma_j=\sigma_j\sigma_i & (|i-j|\ge 2)\\ \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} & (i=1,\ldots,n-2)\end{array}\right\rangle.$$

Where σ_i denotes a crossing between the strings occupying positions i and $i+1$, such that the string in position i crosses (in the up to down direction) over the other, analogously σ_i^{-1}, the algebraic inverse of σ_i, denotes the crossing between the same strings, but in the negative sense, i.e., the string in position i crosses under the other. A *positive braid* is a braid with only positive crossings. A simple braid is a positive braid such that each two strings cross each other at most once. So there is a canonical bijection between the permutation group Σ_n and the set S_n, of simple braids with n strings, which associates to each permutation π, the braid b_π, where each point i is connected by a straight line to $\pi(i)$, keeping all the crossings positive.

Let X be a periodic sequence with least period k and let $\varphi\in\Sigma_k$ be the permutation that associates to each i the position occupied by $s^i(X)$ in the lexicographic ordering of the k-tuple $(s(X),\ldots,s^k(X))$ $(s^k(X)=X)$. Define $\pi\in\Sigma_k$ to be the permutation given by $\pi(\varphi(i))=\varphi(i \mod k+1)$, i.e., $\pi(i)=\varphi(\varphi^{-1}(i) \mod k+1)$. We associate to π the corresponding simple braid $b_\pi\in B_k$ and call it the *Lorenz braid* associated to X. Since X is periodic, this braid represents a knot, and we call it the *Lorenz knot* associated to X. The same method is valid if we consider a pair of sequences. We obtain in this case a Lorenz braid which represents a *Lorenz link*. The Lorenz braid produced in this way is just one possible representative for the respective Lorenz link.

Example: Let $Z=(LRRLRRLRRLR)^\infty$. Hence we have $s^{11}(Z)=Z$, $s(Z)=(RRLRRLRRLRL)^\infty$, $s^2(Z)=(RLRRLRRLRLR)^\infty$,
$s^3(Z)=(LRRLRRLRLRR)^\infty$, $s^4(Z)=(RRLRRLRLRRL)^\infty,\cdots$. Now after lexicographic reordering the $s^i(Z)$ we obtain $s^9(Z)<s^6(Z)<s^3(Z)<$

$s^{11}(Z) < s^8(Z) < s^5(Z) < s^2(Z) < s^{10}(Z) < s^7(Z) < s^4(Z) < s^1(Z)$ and $\varphi = (1, 11, 4, 10, 8, 5, 6, 2, 7, 9)$ written as a disjoint cycle. Finally we obtain $\pi = (1, 8, 4, 11, 7, 3, 10, 6, 2, 9, 5)$ and

$$b_\pi = \sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7$$

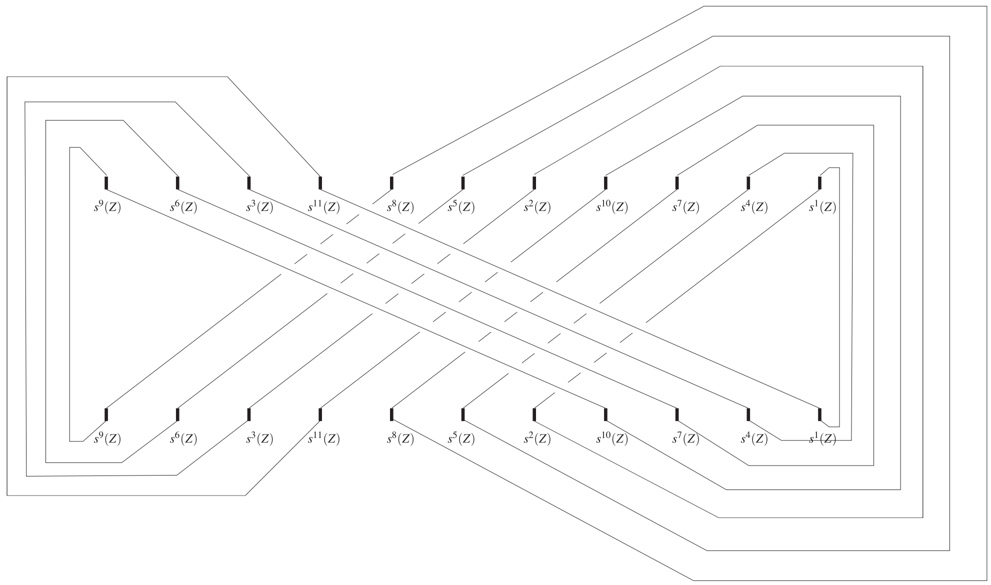

Fig. 1. The Lorenz knot associated to $Z = (LRRLRRLRRLR)^\infty$

Given an admissible pair (X, Y) of symbolic sequences, there is another way of producing a braid representing the Lorenz link associated to (X, Y). This method was developed by Birman and Williams (see [2]) and contains an explicit formula to compute a reduced braid which represents (as a closed braid) the same lorenz knot.

The *trip number*, t, of a finite sequence X, is the number of syllables in X, a syllable being a maximal subword of X, of the form $L^a R^b$. The trip number of an admissible pair of sequences is the sum of the two trip numbers of the two sequences.

Birman and Williams conjectured in [2] that, for the case of a Lorenz knot τ, $b(\tau) = t(\tau)$, where $b(\tau)$ is the braid index of the finite sequence associated to τ. In [10], following a result obtained by Franks and Williams in [5], Waddington observed that this conjecture is true.

The Birman-Williams formula is obtained in the following way. Let $\pi \in S_n$ be a Lorenz permutation. So we construct the Birman-Williams braid $b_{BW}(\pi)$ (or simply b_{BW} if there is no risk of confusion) associated to π in the following way:

$$b_{BW}(\pi) = \Delta_n^2 \prod_{i=1}^{t-1} (\sigma_1 \cdots \sigma_i)^{n_i} \prod_{i=t-1}^{1} (\sigma_{t-1} \cdots \sigma_i)^{m_{t-i}}$$

where t is the trip number, the exponents are given by

$$n_i = card\{j \text{ such that } \pi(j) - j = i+1 \text{ and } \pi(j) < \pi^2(j)\}$$

$$m_i = card\{j \text{ such that } j - \pi(j) = i+1 \text{ and } \pi(j) > \pi^2(j)\}$$

and $\Delta_n \in B_n$ is the simple braid such that each two strings cross each other exactly once. It can be written, in terms of generators, in the following way:

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots \sigma_1\sigma_2\sigma_1.$$

Example: Let $Z = LRRLRRLRRLR$. We have $\pi = (8, 9, 10, 11, 1, 2, 3, 4, 5, 6, 7) \in S_{11}$ moreover the trip number is 4, $n_i = 0$ for $i = 1, \dots, 3$, $m_i = 0$ for $i = 1, \dots, 2$ and $m_3 = 3$. Finally we have

$$b_{BW}(\pi) = \Delta_4^2(\sigma_3\sigma_2\sigma_1)^3.$$

Remark 1 *Since the trip number equals the braid index, then Birman-Williams method produces a braid with a minimal number of strands.*

The *multivariable Alexander polynomial* is a knot invariant that can be described in several different ways. We will follow [8]. In this case the Alexander polynomial is computed as a characteristic polynomial of the reduced Burau linear representation of the closed braid that represents our knot or link. In order to compute the multivariable Alexander polynomial of a braid $\beta \in B_n$ (with n strands), we first compute the Burau colored matrix. First we consider the $(n-1)\times(n-1)$ matrices $\overline{C_i}(a)$ that are equal to the identity matrix except on the i-th line:

$$\overline{C}_i(a) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & a & -a & 1 & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

with $\overline{C}_1(a)$ and $\overline{C}_{n-1}(a)$ properly truncated. We color each strand of β with labels $t_1, \ldots, t_n$. Now we replace in β each appearance of $\sigma_i^{\pm 1}$ by $\overline{C}_i^{\pm 1}(t_j)$, where t_j is the label of the string passing under. Multiplying these matrices we obtain a colored representation, $\overline{C}_\beta(t_1, \ldots, t_n)$ of the braid β. We compute now the Alexander polynomial $\dfrac{\det(I - \overline{C}_\beta(t_1, \ldots, t_n))}{(1 - t_1 \cdots t_n)}$ and set $t_i = t_j$ if the i-th and the j-th strands belong to the same link component.

Example: Let $Z = LRRLRRLRRLR$, the associated BW braid is:

$$b_{BW}(Z) = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 \in B_4$$

The reduced Burau representation, in this case, is an application $\overline{C}$ from B_4 to $GL(3)$ with coefficients in $\mathbb{Q}[x, \frac{1}{x}]$ given by:

$$\overline{C_1}(x) = \begin{bmatrix} -x & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overline{C_2}(x) = \begin{bmatrix} 1 & 0 & 0 \\ x & -x & 1 \\ 0 & 0 & 1 \end{bmatrix} \overline{C_3}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & -x \end{bmatrix}$$

So we have:

$$\overline{C}(b_{BW}(Z)) = \begin{bmatrix} -x^7 & x^7 & 0 \\ -x^7 & 0 & x^7 \\ -x^7 & 0 & 0 \end{bmatrix}$$

We compute now

$$\det(I - \overline{C}(b_{BW}(Z))) = 1 + x^7 + x^{14} + x^{21}.$$

This polynomial factors in:

$$(x+1)\left(x^2+1\right)\left(x^6-x^5+x^4-x^3+x^2-x+1\right)\left(x^{12}-x^{10}+x^8-x^6+x^4-x^2+1\right).$$

Now we divide by $(1-x^4)$ and multiply by $(1-x)$ and obtain:

$$P_Z(x) = \left(x^6-x^5+x^4-x^3+x^2-x+1\right)\left(x^{12}-x^{10}+x^8-x^6+x^4-x^2+1\right).$$

We will test how the different methods for producing a braid affect the computation of the multivariable Alexander polinomial. To do that we will concentrate on irreducible sequences, since it is expectable that the invariants of Lorenz knots and Lorenz links can be obtained from the invariants of their irreducible factors (see [4]).

We define the $*$-product between a pair of finite sequences $(X,Y) \in \Sigma \times \Sigma$, and a sequence $U \in \Sigma$ as

$$(X,Y) * U = \overline{U}_0 \overline{U}_1 \cdots \overline{U}_{|U|-1} 0,$$

where

$$\overline{U}_i = \begin{cases} X_0 \cdots X_{|X|-1} & if\ U_i = L \\ Y_0 \cdots Y_{|Y|-1} & if\ U_i = R \end{cases}.$$

Now we define the $*$-product between two pairs of sequences, $(X,Y),(U,T) \in \Sigma \times \Sigma$, X and Y finite, as

$$(X,Y) * (U,T) = ((X,Y) * U, (X,Y) * T).$$

A sequence is said to be reducible if it can be written as the $*$-product of one admissible pair and one sequence, otherwise it is said to be irreducible. One admissible pair is said to be reducible if it can be written as the $*$-product of two admissible pairs, otherwise it is said to be irreducible.

A Lorenz map is renormalizable if and only if its kneading invariant is reducible, moreover, the itineraries of points in the renormalization interval are all reducible of type $(X,Y) * Z$ where (X,Y) is the first factor of the decomposition of the kneading invariant (see [6] and [4]).

The irreducible Lorenz sequences are easy to construct using the symbolic Farey tree (see [9]).

We construct each level of the Farey tree of maximal Lorenz sequences (i. e. those that are the left elements of admissible pairs) recursively, concatenating the neighbors in the previous level always putting the biggest (lexicographically) word started with an L on the left. So we have:

Level 0: $L < R$

Level 1: $L < LR < R$

Level 2: $L < LRL < LR < LRR < R$

Level 3: $L < LRLL < LRL < LRLRL < LR < LRRLR < LRR < LRRR < R$

$\vdots$

Analogously we construct the Farey tree of minimal Lorenz sequences (i. e. those that are the right elements of admissible pairs) recursively, concatenating the neighbors in the previous level always putting the smallest word started with an R on the left.

3 Comparison of algorithms

We start by presenting the algorithms that where implemented in Maple language[3].

(1) **The Lorenz braid algorithm (L)**
INPUT: One admissible pair of sequences $L = (Z_1, Z_2)$.
  (a) Compute the Lorenz braid b_L associated to L.
  (b) Compute $p(x_1, x_2)$, the multivariable Alexander polynomial associated to the closed braid b_L.
OUTPUT: The multivariable Alexander polynomial $p_{(Z_1,Z_2)}(x_1, x_2)$.

(2) **The Birman-Williams reduced braid algorithm (BW)**
INPUT: One admissible pair of sequences $L = (Z_1, Z_2)$.
  (a) Compute the reduced braid b_{BW} associated to L.
  (b) Compute $p(x_1, x_2)$, the multivariable Alexander polynomial associated to the closed braid b_L.

[3] For the computation of the Alexander polynomial we use an implementation made by Julian Hodgson that is usually available on the Liverpool's knot-theory website, http:\\www.liv.ac.uk\PureMaths\knots.html

OUTPUT: The multivariable Alexander polynomial $p_{(Z_1,Z_2)}(x_1, x_2)$.

In order to test the previous algorithms we proceeded in the following way. We generated pairs of irreducible lorenz sequences up to 6 levels in the Farey tree. We also generated 380 irreducible sequences (up to level 9 in the Farey tree). We tested the two algorithms presented above for the sequences and the pairs of sequences generated. We computed the running time of the main parts of the algorithms (computation of the braids and of the Alexander polynomial). The results obtained are resumed in tables 1 and 2 below:

Table 1
Comparison of algorithms L and BW, for admissible pairs of Lorenz sequences.

TN	Nsl	Ntn	LBBW	LBL	ATABW	ATAL	ML1	ML2	AT1	AT2
2	9.455	66	4.121	10.576	0.024545	0.237288	12	23	0.712	1.667
3	12.727	88	13.000	21.727	0.060898	0.555193	27	41	0.716	0.864
4	15.399	168	25.071	35.464	0.155637	1.424536	54	71	0.750	1.125
5	17.652	184	37.761	49.413	0.337609	2.627560	75	94	4.489	8.658
6	19.752	274	53.241	65.993	0.569161	4.684679	110	131	2.843	5.642
7	22.333	192	71.229	85.563	1.084411	8.587620	121	142	1.542	2.688
8	23.776	268	88.283	103.060	1.489090	39.740978	154	175	11.007	41.675
9	25.647	204	107.922	123.569	2.293015	23.824681	189	212	2.466	2.078
10	28.126	206	133.379	150.505	19.691165	28.966888	234	259	24.015	9.913
11	29.929	112	156.571	174.500	5.126312	41.439446	201	223	4.723	49.696
12	31.750	128	184.875	203.625	18.020656	68.308102	244	268	35.938	3.422
13	34.167	48	217.833	238.000	18.394167	83.737479	273	298	18.583	6.146
14	35.000	36	247.333	267.333	51.038361	127.654722	312	335	3.833	5.611
15	37.500	32	282.250	303.750	51.514875	173.252969	345	369	5.813	5.375
16	40.000	10	331.200	354.200	25.189400	171.153400	390	415	6.300	6.200

Remark: The averages are computed between all sequences (or pairs) with the same trip number. Notice that the trip number, denoted by TN, is also the number of strands in the Birman-Williams braid.

Where:

- Nsl, is the average of the number of strands of the Lorenz braid,
- NtnN, is the number of pairs with equal trip number,

- LBBW, is the average of lengths of the Birman-Williams braids for each trip number,
- LBL, is the average of lengths of the Lorenz braids for each trip number,
- ATABW, is the average time, in seconds, for algorithm BW,
- ATAL, is the average time, in seconds, for algorithm L,
- ML1, is the maximal length found for the Birman-Williams braids for each trip number,
- ML2, is the maximal length found for the Lorenz braids for each trip number,
- AT1, is the average time, in miliseconds, for producing the Birman-Williams braids for each trip number,
- AT2, is the average time, in miliseconds, for producing the Lorenz braids for each trip number,

As we can observe the average running times of both algorithms are very different, showing that algorithm BW is faster than algorithm L. The average length of the braids is quite similar. So the difference in the number of strands is a key factor to speed up the computation of the multivariable Alexander polynomials of Lorenz links. This is due to the fact that the algorithm we used to compute the knot invariants lies on the construction of a matrix representation of the braids. This representation is made in the matrix group $GL(n-1)$, where n is the number of strands thus justifying the difference of performance when increasing the number of strands.

4 Comparison of invariants

Using the same set of sequences, we compared the behavior of knot and link invariants, namely the trip number, the genus (to be defined below) and the Alexander polynomial.

The *genus* g of a link L is the genus of M, where M is an orientable surface of minimal genus spanned by L.

From Theorem 1.1.18 of [3], given a link K and a braid representative b_K of the link, we have

$$g(K) = \frac{C - N - u}{2} + 1,$$

where C is the number of crossings in b_K, N the string index and u the number of link components.

The following table shows, for irreducible sequences up to level 9 in the Farey tree, as the trip number increases, how many different sequences with a given trip number, and how many different genus and Alexander polynomials can we find in sequences with a given trip number:

Trip Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Sequences	16	13	22	18	32	10	34	15	20	10	32	10	25	12	6	10
Different Genus	1	7	12	10	18	6	20	9	12	6	20	6	17	8	4	8
Different Polynomials	1	7	12	10	18	6	20	9	12	6	20	6	17	8	4	8

Trip Number	17	18	19	20	21	22	23	24	25	26	27	29	30	31	34
Sequences	17	12	16	2	6	4	8	2	6	4	4	6	2	4	2
Different Genus	13	8	12	2	4	4	8	2	6	4	4	6	2	4	2
Different Polynomials	13	8	12	2	4	4	8	2	6	4	4	6	2	4	2

We did not find any two sequences with the same Alexander polynomial and different trip number or genus. Moreover, we did not find any two sequences with the same pair (trip number, genus) and different Alexander polynomials. This is totally supported by the results in the previous table, where we can see that, up to level 9 in the Farey tree, for each fixed trip number we have exactly the same number of different genus and different Alexander Polynomials. This experimental set of results led us to conjecture the following:

Conjecture: *Let K_1, K_2 be Lorenz knots. Then $(t(K_1), g(K_1)) = (t(K_2), g(K_2))$ iff $P_{K_1} = P_{K_2}$, where P_{K_i} is the Alexander polynomial of K_i for $i = 1, 2$.*

The following table shows the same as the previous one, for admissible pairs of irreducible sequences up to level 6 in the Farey tree:

Trip Number	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Pairs of Sequences	199	264	504	552	822	576	804	612	618	336	384	144	108	96	30
Different Genus	6	10	16	21	26	30	35	33	38	31	33	22	17	15	6
Different Polynomials	6	14	38	58	88	96	100	84	106	56	64	24	20	16	6

As we can see in the next example, the previous conjecture is not true for Lorenz links, moreover, the invariants Trip Number and Genus cannot be compared. However the Alexander polynomial seems to be a thinner invariant than the trip number and genus for Lorenz links.

Example: Consider the Lorenz links $Z_1 = (LRRLRRLR, RLLRLRL)$, $Z_2 = (LRRRRR, RLRLRRLRRLR)$, $Z_3 = (LRRR, RLLRLLRLRLLRL)$, and $Z_4 = (LRRLRLRLR, RLLLRLL)$. The braids obtained are:

$$
\begin{aligned}
b_L(Z_1) &= \sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11} \\
&\quad \sigma_{12}\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_1\sigma_2\sigma_3 \\
b_{BW}(Z_1) &= \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3 \\
&\quad \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_5\sigma_4\sigma_5\sigma_4 \\
b_L(Z_2) &= \sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}\sigma_{15}\sigma_{16}\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12} \\
&\quad \sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8 \\
b_{BW}(Z_2) &= \sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1 \\
&\quad \sigma_4\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1 \\
b_L(Z_3) &= \sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}\sigma_{15}\sigma_{16}\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12} \\
&\quad \sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_2 \\
&\quad \sigma_3\sigma_4\sigma_5\sigma_6\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \\
b_{BW}(Z_3) &= \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4 \\
&\quad \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_4 \\
b_L(Z_4) &= \sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}\sigma_{15}\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{14}\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13} \\
&\quad \sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_3\sigma_4\sigma_2\sigma_3\sigma_1\sigma_2 \\
b_{BW}(Z_4) &= \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4 \\
&\quad \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_1\sigma_5\sigma_4\sigma_3
\end{aligned}
$$

The invariants are:

	t	g	$P_Z(x_1, x_2)$
Z_1	6	15	$x_1^3 x_2^5 + 1 + x_1^6 x_2^{10}$
Z_2	5	15	${x_1}^{24}{x_2}^6 + {x_1}^{20}{x_2}^5 + {x_1}^{16}{x_2}^4 + {x_1}^{12}{x_2}^3 + {x_1}^8{x_2}^2 + {x_1}^4 x_2 + 1$
Z_3	6	18	$-{x_1}^{32}{x_2}^4 - {x_1}^{24}{x_2}^3 - {x_1}^{16}{x_2}^2 - {x_1}^8 x_2 - 1$
Z_4	6	15	$-(x_2^{10}x_1^4 + 1)(x_1^5 x_2^4 + 1)(x_2^5 x_1^2 + 1)$

In resume, we have that:

$g(Z_1) = g(Z_2)$ and $t(Z_1) \neq t(Z_2)$; $t(Z_1) = t(Z_3)$ and $g(Z_1) \neq g(Z_3)$; $g(Z_1) = g(Z_4)$, $t(Z_1) = t(Z_4)$ and $P_{Z_4}(x_1, x_2) \neq P_{Z_1}(x_1, x_2)$.

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Table 2
Comparison of algorithms L and BW, for Lorenz sequences

TN	Nsl	Ntn	LBBW	LBL	ATABW	ATAL	ML1	ML2	AT1	AT2
2	10.538	13	8.539	16.077	0.033615	0.119923	15	29	0.000	0.000
3	14.318	22	22.636	32.955	0.052500	0.461682	40	59	0.000	0.727
4	17.111	18	39.333	51.444	0.102222	1.007056	69	91	2.667	4.333
5	20.313	32	61.250	75.563	0.193000	2.260531	112	139	2.406	1.438
6	20.273	11	71.364	84.636	0.233818	2.223727	115	137	4.273	7.091
7	25.278	36	109.667	126.944	0.477667	6.606194	198	230	3.944	5.278
8	28.000	18	140.000	159.000	0.985389	12.044889	259	295	1.722	3.444
9	29.700	20	165.600	185.300	0.865150	13.314600	272	305	2.300	3.850
10	33.000	10	207.000	229.000	1.638100	23.007200	333	369	3.200	4.700
11	34.833	36	238.333	261.167	2.275167	156.189722	410	450	7.861	6.444
12	39.600	10	303.600	330.200	3.435200	53.913800	473	515	9.400	24.800
13	38.618	34	307.412	332.029	5.165500	54.026941	564	610	11.559	6.471
14	42.000	16	364.000	391.000	7.252000	75.506688	507	545	3.813	10.688
15	45.000	8	420.000	449.000	7.959750	106.394500	574	614	23.500	4.000
16	44.000	16	420.000	447.000	14.077062	190.416812	615	655	11.563	22.375
17	47.357	28	485.714	515.071	13.512607	142.906393	736	781	7.786	10.500
18	54.000	16	612.000	647.000	21.292062	256.758188	833	881	12.750	11.750
19	53.833	24	627.000	660.833	23.868292	268.845208	900	949	12.250	10.958
20	50.000	4	570.000	599.000	21.185250	164.779500	589	619	8.000	11.750
21	63.000	8	840.000	881.000	45.511375	568.995875	1100	1154	15.250	36.750
22	55.000	8	693.000	725.000	35.591750	274.823375	819	857	27.375	15.625
23	57.500	16	759.000	792.500	46.562562	341.537688	902	942	20.750	12.813
24	60.000	4	828.000	863.000	65.220250	435.235250	943	983	15.500	15.500
25	62.500	12	900.000	936.500	373.666667	517.122667	1032	1074	14.417	24.667
26	65.000	8	975.000	1013.000	135.947125	685.922625	1125	1169	17.750	17.625
27	67.500	8	1053.000	1092.500	80.269875	762.926500	1196	1241	23.375	17.750
29	72.500	12	1218.000	1260.500	174.210083	1083.560333	1400	1449	42.917	46.417
30	75.000	4	1305.000	1349.000	205.528000	1317.400750	1421	1469	23.250	23.250
31	77.500	8	1395.000	1440.500	265.024750	1551.621000	1500	1549	19.750	23.250
34	85.000	4	1683.000	1733.000	468.665750	2560.310500	1815	1869	27.500	27.500